THE MATHEMATICAL GAZETTE

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D. General theory of functions and its application to algebraical and circular functions; infinite series and expansions, especially infinite products and continued fractions considered from the algebraical point of view; Bernoulli's numbers; spherical and analogous functions.

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SOME NEW THEOREMS IN GEOMETRY OF A SURFACE.

BY PROF. C. E. WEATHERBURN, D.Sc.

I. FAMILY OF CURVES ON A SURFACE.

1. The elementary properties of generators of a ruled surface, and the existence of a line of striction when the surface is skew, are well known to readers of this journal. We propose to show that many of these properties do not belong exclusively to ruled surfaces; but that a family of curves on any surface possesses a *line of striction* and a *focal curve* or *envelope*, though these are not always real. When the surface is developable, and the "curves" are the generators of one system, the focal curve is the edge of regression. We shall also see that the properties to be established for a family of curves on a surface are analogous to some of the leading properties of congruences of curves in space, the line of striction corresponding to the surface of striction or orthocentric surface,* and the focal curve to the focal surface of the latter.

Consider then a singly infinite family of curves on a given surface. The curvilinear coordinates u, v for points on the surface may be chosen so that v is constant along any one curve of the family, and u is the distance measured along a curve from a fixed line $u=0$. The family of curves considered are then the parametric curves $v=\text{const.}$ The position vector $\mathbf{r}$ of a point on the surface is a function of u, v; thus $\mathbf{r}=\mathbf{r}(u, v)$. Let suffixes 1, 2 be used to denote partial differentiations with respect to u and v, so that

$$\mathbf{r}_1=\frac{\partial \mathbf{r}}{\partial u}, \quad \mathbf{r}_2=\frac{\partial \mathbf{r}}{\partial v}, \quad \mathbf{r}_{12}=\frac{\partial^2 \mathbf{r}}{\partial u\, \partial v},$$

and so on. Also let E, F, G, H denote as usual the magnitudes introduced by Gauss, and defined by

$$E=\mathbf{r}_1^2, \quad F=\mathbf{r}_1 \cdot \mathbf{r}_2, \quad G=\mathbf{r}_2^2, \quad H^2=EG-F^2.$$

With the above choice of u, $\mathbf{r}_1$ is a unit vector and $E=1$ identically.

If now consecutive curves of the family, corresponding to the parameter

* See a recent paper by the author, *On Congruences of Curves*, § 6. The writer of the present paper has suggested to the author that the term "surface of striction" may be preferable to the term "orthocentric surface" originally employed.

values v, $v+dv$, intersect, a point $\mathbf{r}(u, v)$ on the former must be identical with a point $\mathbf{r}(u+du, v+dv)$ on the latter. Thus,

$$\mathbf{r}(u, v)=\mathbf{r}(u+du, v+dv) \quad \ldots\ldots(1)$$
$$=\mathbf{r}(u, v)+\mathbf{r}_1 du+\mathbf{r}_2 dv,$$

and therefore
$$\mathbf{r}_1 du+\mathbf{r}_2 dv=0.$$

But this is possible only where $\mathbf{r}_1$ is parallel to $\mathbf{r}_2$, and therefore $\mathbf{r}_1 \times \mathbf{r}_2=0$. Now this cross product has the value * $H\mathbf{n}$, where $\mathbf{n}$ is the unit vector normal to the surface. Hence consecutive curves can intersect only where $H=0$. And, conversely, where this relation is satisfied there exists a value of the quotient du/dv for which (1) holds. *The locus of the ultimate points of intersection of consecutive curves may be called the envelope or focal curve of the family. Its equation is* †

$$H=0, \quad \text{or} \quad EG=F^2. \quad \ldots\ldots(2)$$

As in the case of plane curves each member touches the envelope; and the points of contact may be called the *foci* of the curve. The number of these on any curve is the number of roots of (2) regarded as an equation in u. Some (or all) of the roots may be imaginary, as in the case of a family of generators on a skew surface.

2. *Line of Striction.* Consider next the possibility of a normal to a curve of the family, being normal also to a consecutive curve. Let it meet the one curve at the point $\mathbf{r}$ corresponding to the parameter values (u, v), and the other at the point $\mathbf{r}+d\mathbf{r}$ corresponding to the values $(u+du, v+dv)$. Then $d\mathbf{r}$ is perpendicular to the tangents to the curves at these points, which are parallel to $\mathbf{r}_1$ and $\mathbf{r}_1+d\mathbf{r}_1$. Thus

$$d\mathbf{r}\cdot\mathbf{r}_1=0, \quad d\mathbf{r}\cdot(\mathbf{r}_1+d\mathbf{r}_1)=0,$$

from which it follows that

$$(\mathbf{r}_1 du+\mathbf{r}_2 dv)\cdot\mathbf{r}_1=0$$

and
$$(\mathbf{r}_1 du+\mathbf{r}_2 dv)\cdot(\mathbf{r}_{11} du+\mathbf{r}_{12} dv)=0.$$

The first of these gives $du/dv=-F/E=-F$; and on substituting this value of du/dv in the second, remembering that $E=1$, we obtain a result that may be expressed

$$G_1-2FF_1=0,$$

that is to say,
$$H\frac{\partial H}{\partial u}=0.$$

Now $H=0$ is the equation of the focal curve which has already been considered. The alternative $H_1=0$ is the required locus of points at which an arc perpendicular to the curve through that point is also perpendicular to the consecutive curve. Such points correspond to the "central points" of generators of a skew surface. They may be called the *points of striction* of the curves, and their locus the *line of striction* of the family. Thus the equation of the line of striction ‡ is $H_1=0$. Now the unit vector tangent to the curve $v=$const. is $\mathbf{r}_1$, and the two-parametric divergence § of this vector has the value

$$\frac{1}{H}\frac{\partial H}{\partial u}=\operatorname{div}\mathbf{r}_1.$$

---

* Cf. the author's *Differential Geometry*, Art. 25.

† This is also the focal curve of the family $u=$const., and of the family $\phi(u, v)=$const.

‡ More generally, when E is not equal to unity, the line of striction of the parametric curves $v=$const. is given by

$$\frac{\partial}{\partial u}\left(\frac{H}{\sqrt{E}}\right)=0.$$

§ All the differential invariants (grad., div., curl, etc.) of this paper are the two-parametric invariants introduced by the author in a recent paper, "On Differential Invariants in Geometry of Surfaces, etc.," *Quarterly Journal of Math.*, 1925.

Hence, quite independently of any choice of parametric curves, we may state our theorem as follows:

Given a one-parameter family of curves on a surface, with $\mathbf{t}$ *as the unit tangent, the equation of the line of striction of the family is div* $\mathbf{t}=0$.

The function div $\mathbf{t}$ may very appropriately be called the *divergence* of the family. Hence the line of striction is the locus of points at which the divergence of the family vanishes. The appropriateness of the term is otherwise apparent in the case of generators of a ruled surface. For these converge to the line of striction and thence diverge; so that the divergence changes sign from negative to positive at the line of striction.

Or further, since the geodesic curvature of a curve is equal to the divergence of the unit vector normal to the curve and tangential to the surface,* the above theorem may be expressed in the form:

The line of striction of a family of curves is the locus of points at which the geodesic curvature of their orthogonal trajectories is zero.

In connection with *ruled surfaces* it is often loosely stated that the edge of regression of a developable corresponds to the line of striction of a skew surface. The curve known as the edge of regression has a dual nature which may be explained as follows. On any generator of a skew surface the foci are two imaginary points equidistant from the point of striction. As the specific curvature K of the surface tends to zero, these points tend to coincidence. In the limit when $K=0$ the surface is developable, and the foci become two real points coincident with the point of striction.† Thus *on a developable surface the focal curve coincides with the line of striction in the curve known as the edge of regression.*

3. *Family of Geodesics.* Let the family of curves under consideration be a family of geodesics. We may choose these for the parametric curves $v=$const., and their orthogonal trajectories for the curves $u=$const. Then if $\mathbf{a}$ and $\mathbf{b}$ denote the unit tangents to the parametric curves, div $\mathbf{b}$ vanishes identically, and div $\mathbf{a}=0$ is the equation of the line of striction of the family of geodesics.

Consider any other curve cutting the geodesics at a (variable) angle θ. The unit tangent $\mathbf{t}$ to this curve is then

$$\mathbf{t}=\mathbf{a}\cos\theta+\mathbf{b}\sin\theta,$$

and the unit vector perpendicular to this, and tangential to the surface, is $\mathbf{b}\cos\theta-\mathbf{a}\sin\theta$. Hence the geodesic curvature of the curve has the value

$$\begin{aligned}\operatorname{div}(\mathbf{b}\cos\theta-\mathbf{a}\sin\theta)&=-\sin\theta\operatorname{div}\mathbf{a}-\sin\theta\,\mathbf{b}\cdot\nabla\theta-\cos\theta\,\mathbf{a}\cdot\nabla\theta\ddagger\\&=-\sin\theta\operatorname{div}\mathbf{a}-\mathbf{t}\cdot\nabla\theta.\end{aligned}$$

If the first member of this equation vanishes, the curve is a geodesic. If div $\mathbf{a}$ vanishes, the point is on the line of striction; and if the last term vanishes, the curve cuts the family of geodesics at a constant angle. Hence the theorem:

If a curve is drawn on a surface so as to cut a family of geodesics, then, provided it has two of the following properties, it will also have the third: (*a*) *that it is a geodesic,* (*b*) *that it is the line of striction of the family of geodesics,* (*c*) *that it cuts the family at a constant angle.*

This is a generalisation of Bonnet's theorem § on the generators of a ruled surface.

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* *Ibid.* Art. 8.

† All this is easily verified from the value $H^2=a^2u^2+2bu+\sin^2\theta$, where a, b, θ are independent of u. See the author's *Differential Geometry*, Chap. VII., Art. 70.

‡ Using the expansion formula $\operatorname{div}\phi\mathbf{s}=\phi\operatorname{div}\mathbf{s}+\mathbf{s}.\nabla\phi$, and the elementary results

$$\nabla\sin\theta=\cos\theta\,\nabla\theta,\text{ etc.}$$

§ See the author's *Differential Geometry*, Art. 72.

4. The preceding theorems are connected with the divergence of the unit tangent to the family of curves. Consider now the *curl* of this vector. As in the last paragraph, let the family of curves be chosen as the parametric curves $v=\text{const.}$, and their orthogonal trajectories as the curves $u=\text{const.}$ Then the two-parametric curl of the unit tangent $\mathbf{a}$ to the family of curves is given by *

$$\text{curl}\,\mathbf{a}=\frac{M}{\sqrt{EG}}\,\mathbf{a}-\frac{L}{E}\,\mathbf{b}-\frac{E_2}{2E\sqrt{G}}\,\mathbf{n},$$

where L, M, N are the second order magnitudes for the surface, defined by

$$L=\mathbf{n}\cdot\mathbf{r}_{11},\quad M=\mathbf{n}\cdot\mathbf{r}_{12},\quad N=\mathbf{n}\cdot\mathbf{r}_{22}.$$

Thus

$$\mathbf{a}\cdot\text{curl}\,\mathbf{a}=\frac{M}{\sqrt{EG}},$$

which is equal to the "geodesic torsion" † of the curve $v=\text{const.}$, that is to say, the torsion of the geodesic tangent. Similarly

$$\mathbf{n}\cdot\text{curl}\,\mathbf{a}=-\frac{E_2}{2E\sqrt{G}},$$

which is the geodesic curvature of the curve ‡ $v=\text{const.}$ Lastly,

$$(\mathbf{a}\times\mathbf{n})\cdot\text{curl}\,\mathbf{a}=-\mathbf{b}\cdot\text{curl}\,\mathbf{a}=\frac{L}{E},$$

which is the normal curvature of the surface in the direction of the curve $v=\text{const.}$ Hence the theorem:

If $\mathbf{t}$ *is the unit tangent at any point to the member of a family of curves drawn on a surface, the "geodesic torsion" of the curve is* $\mathbf{t}\cdot curl\,\mathbf{t}$, *its geodesic curvature is* $\mathbf{n}\cdot curl\,\mathbf{t}$, *and the normal curvature of the surface in the direction of the curve is* $\mathbf{t}\times\mathbf{n}\cdot curl\,\mathbf{t}$.

Then, since the geodesic torsion vanishes for a line of curvature, the geodesic curvature for a geodesic, and the normal curvature in the direction of an asymptotic line, we have as a corollary:

A family of curves with unit tangent $\mathbf{t}$ *will be lines of curvature if* $\mathbf{t}\cdot curl\,\mathbf{t}$ *vanishes identically; they will be geodesics if* $\mathbf{n}\cdot curl\,\mathbf{t}$ *is zero, and they will be asymptotic lines if* $\mathbf{t}\times\mathbf{n}\cdot curl\,\mathbf{t}$ *is zero.*

If $\mathbf{b}$ is the unit tangent to the curve $u=\text{const.}$, we have similarly

$$\text{curl}\,\mathbf{b}=\frac{N}{G}\,\mathbf{a}-\frac{M}{\sqrt{EG}}\,\mathbf{b}+\frac{G_1}{2G\sqrt{E}}\,\mathbf{n}.$$

Hence

$$\mathbf{n}\cdot\text{curl}\,\mathbf{a}\times\text{curl}\,\mathbf{b}=\frac{LN-M^2}{EG}=K.$$

Thus the *specific curvature* of the surface is given by the formula

$$K=[\mathbf{n},\ \text{curl}\,\mathbf{a},\ \text{curl}\,\mathbf{b}],$$

which we have already proved elsewhere.§

II. ORTHOGONAL SYSTEMS OF CURVES.

5. One of the best known theorems in Geometry of Surfaces is that of Dupin, which states that the sum of the normal curvatures of a surface in any two perpendicular directions is invariant and equal to the mean curvature of the surface. We propose to show that this is only one aspect of a more

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* *On Differential Invariants in Geometry of Surfaces, etc.*, Art. 11 (36).
† *Differential Geometry*, Art. 49. ‡ *Ibid.* Arts. 53-55.
§ *On Differential Invariants, etc.*, Art. 11 (37).

comprehensive theorem dealing with the curvature of orthogonal systems of curves drawn on the surface—a theorem specifying both the mean curvature and the specific curvature. We define the vector curvature * of a curve as the arc-rate of change of the unit tangent. This quantity has the magnitude of the (circular) curvature, and the direction of the principal normal. If then we consider any orthogonal system of curves drawn on the surface, and resolve their vector curvatures into components normal to the surface and tangential to it, the theorem to be proved may be stated:

The sum of the vector curvatures of the two curves of an orthogonal system through any point, has a normal component whose magnitude is equal to the mean curvature of the surface, and a tangential component whose divergence is equal to the specific curvature at that point.

The normal component of this vector curvature is then an absolute invariant, being the same for all orthogonal systems. This is substantially Dupin's theorem. The tangential component is not an absolute invariant; but it possesses an invariant divergence. If $\mathbf{n}$ is the unit vector normal to the surface, the behaviour of this tangential component may be expressed as follows:

The vector curvature is the same for orthogonal systems that cut each other at a constant angle. If, however, the inclination θ of one system to the other is variable, their curvatures differ by the tangential vector curl $(\theta\mathbf{n})$, whose divergence vanishes identically.

To prove the above theorems, let us take the orthogonal system for parametric curves and, as before, let $\mathbf{a}$ and $\mathbf{b}$ denote their unit tangents. Then

$$\mathbf{a}=\frac{\mathbf{r}_1}{\sqrt{E}},\quad \mathbf{b}=\frac{\mathbf{r}_2}{\sqrt{G}},\quad \mathbf{n}=\mathbf{a}\times\mathbf{b}.$$

The vector curvature of the curve $v=$const. is given by †

$$\frac{1}{\sqrt{E}}\frac{\partial\mathbf{a}}{\partial u}=\frac{L}{E}\mathbf{n}-\frac{E_2}{2E\sqrt{G}}\mathbf{b}=\frac{L}{E}\mathbf{n}-\mathbf{b}\operatorname{div}\mathbf{b}.$$

Similarly the vector curvature of the curve $u=$const. is

$$\frac{1}{\sqrt{G}}\frac{\partial\mathbf{b}}{\partial v}=\frac{N}{G}\mathbf{n}-\frac{G_1}{2G\sqrt{E}}\mathbf{a}=\frac{N}{G}\mathbf{n}-\mathbf{a}\operatorname{div}\mathbf{a}.$$

Combining these results, and denoting the mean curvature of the surface by J, we have the sum of the vector curvatures of the orthogonal curves equal to

$$J\mathbf{n}-(\mathbf{a}\operatorname{div}\mathbf{a}+\mathbf{b}\operatorname{div}\mathbf{b}).$$

The component of this normal to the surface is the invariant $J\mathbf{n}$, as stated in the first part of the theorem. The tangential component is the vector $\mathbf{t}$ defined by

$$\begin{aligned}\mathbf{t}&=-(\mathbf{a}\operatorname{div}\mathbf{a}+\mathbf{b}\operatorname{div}\mathbf{b})\\&=-\frac{1}{2}\left(\frac{G_1}{EG}\mathbf{r}_1+\frac{E_2}{EG}\mathbf{r}_2\right),\end{aligned}$$

and the divergence of this vector is ‡

$$\begin{aligned}\operatorname{div}\mathbf{t}&=-\frac{1}{2\sqrt{EG}}\left[\frac{\partial}{\partial u}\left(\frac{G_1}{\sqrt{EG}}\right)+\frac{\partial}{\partial v}\left(\frac{E_2}{\sqrt{EG}}\right)\right]\\&=-\frac{1}{\sqrt{EG}}\left[\frac{\partial}{\partial u}\left(\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u}\right)+\frac{\partial}{\partial v}\left(\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v}\right)\right]\text{ §}\\&=K,\end{aligned}$$

* See *Differential Geometry*, Art. 52. † *Differential Geometry*, Art. 41.

‡ *On Differential Invariants, etc.*, Art. 4 (7).

§ Cf. Bianchi, *Geometria Differenziale*, vol. I. p. 124, § 48.

where K as usual denotes the specific curvature of the surface. This proves the second part of our theorem, that the divergence of the tangential part of the vector curvature is invariant, and equal to the specific curvature. We may also remark that the divergence of the normal component of the vector curvature is given by *

$$\operatorname{div} J\mathbf{n} = -J^2.$$

Hence the theorem :

The divergence of the vector curvature of an orthogonal system of curves on the surface is invariant and equal to $K - J^2$.

6. It remains to examine the tangential component $\mathbf{t}$, and see how it changes with the orthogonal system selected. Consider a second orthogonal system, inclined to the above at an angle θ varying from point to point. The unit tangents to the curves of this system are $(\mathbf{a}\cos\theta + \mathbf{b}\sin\theta)$ and $(\mathbf{b}\cos\theta - \mathbf{a}\sin\theta)$. The tangential component $\mathbf{t}'$ of the vector curvature of this system is then given by

$$\mathbf{t}' = -(\mathbf{a}\cos\theta + \mathbf{b}\sin\theta)\operatorname{div}(\mathbf{a}\cos\theta + \mathbf{b}\sin\theta) - (\mathbf{b}\cos\theta - \mathbf{a}\sin\theta)\operatorname{div}(\mathbf{b}\cos\theta - \mathbf{a}\sin\theta).$$

Expanding this as in § 3, we easily reduce it to the form

$$\begin{aligned}\mathbf{t}' &= -(\mathbf{a}\operatorname{div}\mathbf{a} + \mathbf{b}\operatorname{div}\mathbf{b}) - \mathbf{a}(\mathbf{b}\cdot\nabla\theta) + \mathbf{b}(\mathbf{a}\cdot\nabla\theta)\\ &= \mathbf{t} + \mathbf{n}\times(\mathbf{b}\mathbf{b} + \mathbf{a}\mathbf{a})\cdot\nabla\theta = \mathbf{t} + \mathbf{n}\times\nabla\theta\\ &= \mathbf{t} - \operatorname{curl}(\theta\mathbf{n}).\end{aligned}$$

Thus the vector curvature of the new orthogonal system differs from that of the other by the tangential vector

$$\mathbf{n}\times\nabla\theta = -\operatorname{curl}(\theta\mathbf{n}),$$

whose divergence vanishes identically † ; and we have proved the last part of the theorem stated. If the two orthogonal systems cut at a constant angle θ, then $\nabla\theta$ is zero, and the vector curvatures of the two systems are equal. There is an infinitude of orthogonal systems every two of which cut at a constant angle ; and the vector curvature is the same for all these systems.

Christchurch, N.Z., October, 1925. C. E. WEATHERBURN.

GLEANINGS FAR AND NEAR.

338. Aug. 10, 1664. Abroad to find out one to engrave my tables upon my new sliding rule with silver plates, it being so small that Browne that made it cannot get one to do it. So I got Cocker, the famous writing master, to do it, and I set an hour by him to see him design it all : and strange it is to see him with his natural eyes to cut so small at his first designing it, and read it all over, without any missing, when for my life I could not, with my best skill, read one word, or letter of it ; but it is use. He says that the best light for his life to do a very small thing by (contrary to Chaucer's words to the Sun, " that he should lend his light to them that small seals grave,"), it should be by an artificial light of a candle, set to advantage, as he could do it. I find the fellow, by his discourse, very ingenious : and among other things, a great admirer and well read in the English poets, and undertakes to judge of them all, and that not impertinently.

11th. Comes Cocker with my rule, which he hath engraved to admiration, for goodness and smallness of word ; it cost me 14s. the doing.—Pepys's *Diary*. [*per* the Rev. J. J. Milne.]

* *On Differential Invariants, etc.*, Art. 4 (6).

† *Ibid.* Art. 7 (23).

NOTE ON DEGENERATE CONICS.

By the Rev. J. J. Milne, M.A.

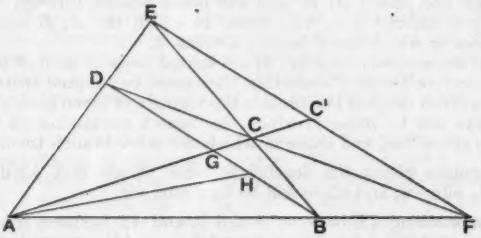
Readers of the article in the October Magazine on "Systems of Conics satisfying four conditions" would notice the important part played by degenerate conics. A little more explanation of these degenerate conics may not be out of place.

To obtain a degenerate conic of the first kind, let the major axis AA' of a central conic remain unchanged whilst the foci S, S' vary their positions. When the foci coincide with the points A, A' we obtain the double line AA', each point of which satisfies the condition $SP \pm S'P = AA'$, the degenerate conic being the two points A, A' and the double line AA', or its complement, according as the positive or the negative sign is taken.

Any straight line through either of the points A, A' (except the line AA') is to be considered as a tangent of the degenerate conic. A degenerate conic of the second kind may be obtained by cutting a double cone by a plane passing through the vertex and two generators of the cone. The degenerate conic in this case consists of the two lines and a double point at their intersection.

When the lines are real, the degenerate conic is the limiting case of a hyperbola which coincides with its asymptotes, the double point being the centre, and lines through this point may be considered as tangents to the degenerate conic.

Other properties are given in Art. 3 of the previous note. Perhaps the subject will be made clearer by a detailed discussion of the five elementary cases (A), (B), (C), (D), (E), with the aid of a diagram giving a geometrical interpretation of the use made of degenerate conics.



(A) When the system passes through the four points A, B, C, D, it is obvious from the figure that there are no degenerate conics of the first kind, whilst there are three of the second kind, viz. (AB, CD), (AC, BD), (AD, BC).

These, by Zeuthen's rule, Art. 11, are to be counted once only.

(B) Given three points A, B, E, and one tangent DF.

There are no degenerate conics of the first kind.

There are three degenerate conics of the second kind, viz.:

(BCE, AC), with its double point at C, where the given tangent cuts BE;

(ADE, BD) with its double point D, where the tangent cuts AE;

and (ABF, EF) with its double point F, where the tangent cuts AB.

By Zeuthen's rule each of these has to be counted twice, for each has a double point at the intersection of the given tangent with the line joining two of the given points, and consists of this latter line and the line joining the double point to the third given point.

(C) Given two points G, H, and two tangents AE, AF.
There is one degenerate conic of the first kind (BD).
There is one degenerate conic of the second kind (AG, AH).
By Zeuthen's rules each of these is to be counted four times. For the degenerate conic (BD) passes through the two given points G, H, and is terminated at the points B, D, where it meets the given straight lines AF, AE.
The degenerate conic (AG, AH) has a double point at A, the intersection of the two given lines AE, AF, and consists of the lines AG, AH which join this point of intersection to the two given points G, H.

(D) Given three tangents AF, FE, EA, and one point C.
There are three degenerate conics of the first kind (AC'), (EB), (FD).
There are no degenerate conics of the second kind.
Now consider the degenerate conic (AC'). It passes through the given point C, and through A the point of intersection of the two given lines AE, AF, and is terminated at the point A, and at the point C' where it meets the third given line EF, and is therefore, by Zeuthen's rule, to be counted twice, and similarly for each of the degenerate conics (EB) and (FD).

(E) Given four tangents AB, BC, CD, DA.
There are three degenerate conics of the first kind, (AC), (BD), (EF).
There are no degenerate conics of the second kind, and there are no repetitions.

The repetitions of the degenerate conics in (B), (C) and (D) are in accordance with Zeuthen's rules, which were obtained from algebraical considerations. Prof. A. Lodge gives the following geometrical reasons for their occurrence.

In (B) it is fairly easy to see that each of the degenerate conics is, in two ways, the limit of a hyperbola; for if the given tangent DF passes between AB and E as in the figure, then of all the hyperbolas of the system, one branch goes through the points A, B, and the other branch through E, and they consist of two distinct sets, viz., those in which the A, B branch touches DC, and those in which the E branch touches it.
The three degenerate conics in (B) are special cases of each of these pairs of sets, and therefore Zeuthen's rule that they must be counted twice is justified.
When the given tangent lies outside the triangle of given points, the two sets of hyperbolas will be those in which the branch containing all three points touches the given line, and those in which the other branch touches it.

(C) The conics giving the degenerate conic of the first kind (BD) must evidently be ellipses, and there will be four sets, viz.:

(1) Those touching AB between A and B, and AD between A and D;
(2) Those touching AB between A and B, and AD produced;
(3) Those touching AB produced, and AD between A and D;
(4) Those touching AB and AD both produced.

In the conics giving the degenerate conic of the second kind (AG, AH), there will be hyperbolas in which (1) the branch containing the given points touches both the given lines, (2) those in which the other branch touches both the lines, and two sets in which one branch touches one of the lines and the other branch touches the other line.

(D) Taking the degenerate conic (AC') as an illustration, we have two sets of ellipses, one of which touches EF between E and C', and the other touches EF between F and C', and each of these sets has (AC') for its degenerate conic, which must therefore be counted twice, and similarly for the other degenerate conics (EB) and (FD).

In (A) no repetitions are required, for we cannot separate the conics into sets for each degenerate conic. Thus (AB, CD) is the degenerate conic of a

system of hyperbolas, in each of which one branch passes through A and B, and the other passes through C and D, and there are no others. Similarly for (AC, BD) and (AD, BC).

Many systems satisfying four conditions which are confined to points and lines are equivalent to one of the above cases; *e.g.* given two points A, B, a line L and its pole P. Join PA cutting L in C, and take A' the fourth harmonic of P, C and A. Then A' is a point on the system. Similarly we can find another point B', and the system is the same as (A), in which four points are given, and its characteristics are therefore (1, 2).

We are now in a position to write down the characteristics of many systems at sight. For noticing that to be given the direction of an asymptote, or a pair of conjugate points is equivalent to being given one condition, viz. a point, and that to be given an asymptote, or a focus, or the pole of a given line is equivalent to two, and a self-conjugate triangle to three conditions, it is easy to see that the following systems are equivalent to (A) and have (1, 2) for their characteristics:

Two points and an asymptote.
Three points and a pair of conjugate points.
One point, a tangent at a given point and a pair of conjugate points.
One point and a self-conjugate triangle.
Two points and the centre.
Three points and the tangent at one of them.
Three points and the pole of the line joining two of them to lie on a given line.

The reader will find it an interesting and useful exercise to write down the systems which are equivalent to the elementary cases (B), (C), (D) and (E), and which therefore have their characteristics the same as these.

I would add that in the *Comptes Rendus* referred to in the previous note, Chasles enunciated 31 propositions without demonstrations. M. T. Lemoyne, in *Les Lieux Géométriques*, has given 88 theorems, to the majority of which are given very neat solutions. The subject of characteristics is also dealt with in *Courbes Géométriques Remarquables* by H. Brocard and T. Lemoyne, in pp. 259-280, where the authors give 83 theorems, covering much the same ground, with hints or full solutions where thought necessary.

The price of *Les Lieux Géométriques* has recently been raised from 10 fr. to 12 fr.

J. J. MILNE.

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339. Charles Lamb to Thomas Manning. February, 1801.—. . . I might also, and did imagine, that you . . . were inventing new properties in a triangle, and for that purpose moulding and squeezing Landlord Crisp's three-cornered beaver into fantastic experimental forms; or that Archimedes was meditating to repulse the French, in case of a Cambridge invasion, by a geometric hurling of folios on their red caps. . . . [Mr. Crisp was Manning's landlord, a barber in St. Mary's Passage.]—*Lamb's Works*, ed. E. V. Lucas, vi. p. 215. [*per* Mr. F. Puryer White.]

340. Charles Lamb to Thomas Manning. April, 1801.—. . . I live at No. 16 Mitre-court Buildings, a pistol-shot off Baron Maseres'. You must introduce me to the Baron. I think we should suit one another mainly. He lives on the ground floor for convenience of the gout; I prefer the attic story for the air! He keeps three footmen and two maids; I have neither maid nor laundress, not caring to be troubled with them! His forte, I understand, is the higher mathematics; my turn, I confess, is more to poetry and the belles lettres. The very antithesis of our characters would make up a harmony. You must bring the Baron and me together. . . . [Francis Maseres (1731-1824) had rooms at 5 King's Bench Walk. To the end he wore a three-cornered hat, a wig and ruffles. See *The Old Benchers*.]—*Lamb's Works*, ed. E. V. Lucas, vi. p. 217. [*per* Mr. F. Puryer White.]

AN INTRODUCTION TO EXPONENTIALS FOR TECHNICAL STUDENTS.

By W. G. Bickley, M.Sc.

The method of introducing e so common in text-books based upon an algebraic proof of the exponential series may be satisfactory for mathematical students, but it does not meet the needs of technical students (especially evening students), for it is impossible, if even desirable, to treat the matter along these lines in a manner at once brief, exact, and comprehensible. Moreover, it suffers from the great disadvantage that the property of e, which is of prime importance in technical applications, namely that e^x is the function appropriate to the expression of the law of growth or decay so common in these—in other words, that e^x is the solution of the differential equation $dy/dx=y$—appears as an afterthought, and not as the fundamentally important property, the *raison d'être* of the inclusion of e in the course and in the engineering formulae. The method outlined below, which has been tried for some few sessions, aims at presenting the differential equation as arising in the attempt to solve a concrete practical problem, and the discovery and identification of two solutions—one a series, the other in terms of indices. By obtaining the series first some idea of the use of non-terminating series, and appreciation of convergence, are gained. To simplify the numerical calculations, the index solution is then found. Here e appears as a limit, and to obtain its numerical value recourse must be had to the series.

As an introduction, growth and decay problems (cooling, decay of current in an inductive circuit, belt friction, compound interest, etc.) are mentioned, and the fact made apparent that the *rate* of growth depends upon the instantaneous value—with the probability that it is in many cases *proportional to* this value. Attention is then focussed upon the friction of a belt round a pulley, and its effect in increasing the tension from the slack to the tight side. A little mechanics leads easily to the equation

$$\frac{dT}{d\theta}=\mu T \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

(T=tension, θ=angle of wrap, and μ=coefficient of friction). The equation is then taken in the simplified form

$$\frac{dy}{dx}=y, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

and it is pointed out that the need is a formula for y which satisfies this. y may *start* at any value, but then its growth is conditioned by the equation. For simplicity, let it start by being unity—if it should have an initial value y_0, we merely have to multiply our solution by y_0, *i.e.*

$$\left.\begin{aligned} y&=1+\ldots. \\ \frac{dy}{dx}&=\quad 1+\ldots. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3a)$$

But that means also that

Now to obtain 1 in dy/dx, there must have been an x in y, and consequently in dy/dx, so that we must have

$$\left.\begin{aligned} y&=1+x+\ldots, \\ \frac{dy}{dx}&=\quad 1+x+\ldots. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3b)$$

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The x in dy/dx necessitates $x^2/2$ in y, and also in dy/dx. Inserting this, the blackboard then reads

$$\left.\begin{aligned} y &= 1 + x + \frac{x^2}{2} + \dots, \\ \frac{dy}{dx} &= 1 + x + \frac{x^2}{2} + \dots. \end{aligned}\right\} \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (3c)$$

$x^2/2$ in dy/dx necessitates $x^3/2{\cdot}3$ in y, etc., and the process of building up y proceeds as indicated by the arrows:

$$\begin{aligned} y &= 1 + x + \frac{x^2}{2} + \frac{x^3}{2.3} + \frac{x^4}{2.3.4} \cdots, \\ \frac{dy}{dx} &= 1 + x + \frac{x^2}{2} + \frac{x^3}{2.3} + \frac{x^4}{2.3.4} \cdots \end{aligned}$$

More generally, $$y = y_0\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots\right). \quad \dots\dots\dots\dots\dots\dots\dots\dots (3e)$$

In the same way, the equation $dy/dx = 3y$ leads to

$$y = y_0\left(1 + 3x + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \dots\right), \quad \dots\dots\dots\dots\dots\dots\dots\dots (4)$$

and the general solution for $dy/dx = ay$ is soon seen—or similarly obtained. Returning to the belt problem, and introducing the tension on the slack side, T_0, we have

$$T = T_0\left(1 + \mu\theta + \frac{(\mu\theta)^2}{2!} + \frac{(\mu\theta)^3}{3!} + \dots\right). \quad \dots\dots\dots\dots\dots\dots\dots\dots (5)$$

A numerical instance is now taken—say $\mu = 0{\cdot}2$, $\theta = \pi$ (angle of wrap 180° !), so that

$$\begin{aligned} T &= T_0\left(1 + {\cdot}6284 + \frac{({\cdot}6284)^2}{2!} + \frac{({\cdot}6284)^3}{3!} + \dots\right) \\ &= 1{\cdot}874\dots T_0. \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (6) \end{aligned}$$

It is now seen that only a few terms of the series are necessary, but the calculations can be laborious.

There is another way round, however. The fundamental equation may be written

$$\delta T = T\,\delta\theta. \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (7)$$

Consequently, $$T + \delta T = T + T\,\delta\theta = T(1 + \delta\theta). \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (8)$$

We may call $(1 + \delta\theta)$ the "growth factor" for the step $\delta\theta$, and, going step by step, starting from T_0, we obtain the *approximate* formula

$$T = T_0(1 + \delta\theta)^n, \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (9)$$

where n is the number of steps. With the same values of μ and θ as before, and with $\theta = 0{\cdot}1$, we find

$$T = T_0(1 + 0{\cdot}02)^{31{\cdot}42\dots} = 1{\cdot}863T_0 \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (10)$$

in fair agreement with (5). The step 0·1 is large, and a better result would be obtained if it were made smaller—if only logarithms to more places were available. Even then finality could not be reached by purely numerical work.

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Further analysis will circumvent this difficulty. Going back to $dy/dx=y$, and using the same method, we have

$$\begin{aligned} y &= y_0(1+\delta x)^n \\ &= y_0(1+\delta x)^{x/\delta x} \\ &= y_0\{(1+\delta x)^{1/\delta x}\}^x. \end{aligned}$$

The quantity in the curly brackets now needs attention—we seek its limit as $\delta x \to 0$. It may be verified that the following values are true:

x	$\frac{1}{4}$	0·1	0·02	0·001
$(1+\delta x)^{1/\delta x}$	2·44...	2·59...	2·69...	2·71...

so that our limit is just over 2·7. Calling this important number e, we can write our solution

$$y = y_0 e^x, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12)$$

although we have still to find an accurate value of e. This can be done by comparing (12) with (3e), when we deduce

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots; \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14)$$

and, putting $x=1$ in this,

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (15)$$

$$= 2{\cdot}71828\ldots .$$

We now have all the important properties of the solution, and among other things,

$$T = T_0 e^\theta,$$

$$e^{ax} = 1 + ax + \frac{(ax)^2}{2!} + \frac{(ax)^3}{3!} + \frac{(ax)^4}{4!} + \ldots .$$

If $dy/dx = ay$, then $y = y_0 e^{ax}$; and conversely, if $y = y_0 e^{ax}$, then

$$dy/dx = ay = ay_0 e^{ax},$$

so that $\frac{d}{dx}(e^{ax}) = ae^{ax}$.

W. G. Bickley.

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341. Charles Lamb to Thomas Manning. December 25, 1815.—. . . Your way of mathematics has already given way to a new method, which after all is I believe the old doctrine of Maclaurin, new-vamped up with what he borrowed of the negative quantity of fluxions from Euler. . . . [In a ragging letter, saying how things have changed since Manning left England. E. V. Lucas has the following note: "Here Lamb surprises the reader by a reasonable remark!"]—*Lamb's Works*, ed. E. V. Lucas, vi. p. 481. [*per* Mr. F. Puryer White.]

342. Manning to his Father. February 12, 1802.—. . . I have had another little subject of disquietude in receiving no answer to a letter I sent to Mr. La Grange, to whom I addressed a question, concerning where I might find the demonstration of a certain Theorem, to which he has never sent any answer—& the Question is important to me—as, *I believe*, no one has ever demonstrated the theorem satisfactorily & I possess in my mind a genuine demonstration.—*Letters of Thomas Manning to Charles Lamb*, ed. G. A. Anderson, 1925. [*per* Mr. F. Puryer White.]

MATHEMATICS FOR GIRLS.

A MEMORANDUM FROM THE GIRLS' SCHOOLS' COMMITTEE OF THE MATHEMATICAL ASSOCIATION.

"THE Report on the Differentiation of Curricula between the Sexes," published by the Board of Education in 1923, suggested that girls' mathematical work was inferior to that of boys. The Girls' School Committee of the Mathemathical Association has been making some enquiries with a view to finding possible remedies.

The Committee deprecates the use of the term "inferiority." That the general standard of attainment is lower in girls' schools than in boys' is not denied, but there are implications in the term "inferiority" which are seen to be unfair when the causes leading to the difference in standard are reviewed. For example, the time given to Mathematics is usually less for girls than boys —roughly amounting to a difference of one period a week. Where boys and girls are given equal opportunities, as in some mixed schools, the consensus of opinion is that very little difference in capacity between boys and girls is shewn in work up to Matriculation standard, but that boys have much more natural interest in Mathematics than girls. The Committee was unanimous that it is not desirable that more time should be given to Mathematics by girls than is now the case. The time allotted, if used to the best advantage, is enough to enable girls to reach a satisfactory examination standard, provided that mathematical papers in a first school examination are sufficiently well graded to meet the needs of pass and credit candidates. Weaker mathematical divisions often fail to pass in Mathematics not from lack of ability to reach a pass standard, but because the questions seem to be set wholly with a view to testing the powers of the "credit" candidate. It seems preferable to award a pass for sound answers to straightforward questions rather than to poor answers to those of a type the pass candidate finds very difficult. The papers of the Oxford and Cambridge Joint Board meet this need, and that an attempt is being made to do so by the Joint Matriculation Board of the Northern Universities is indicated by the specimen papers in Mathematics for 1926 recently sent out.

In addition to the greater facility given by the cumulative effect of longer time, boys' interest in Mathematics is stimulated by their work in Physics. Enquiries made in a large number of girls' schools shew that only 40 % of the mistresses in them teaching Mathematics were qualified to teach elementary Physics. It is desirable that all girls be taught some Physics, and for a girl who is likely to become a mathematical specialist the subject is almost essential. Close co-operation between the Physics and Mathematical staffs would strengthen the work in both subjects, and every mathematical teacher should have some knowledge of Physics.

Another point that emerged from the enquiry was that while in boys' schools the Mathematics of a class was regarded as a whole, very often in girls' schools, especially in middle school divisions, definite periods were assigned to Arithmetic, Algebra and Geometry, and these were not necessarily taken by the same mistress. The former plan is much more likely to produce satisfactory work, and helps to economise time. It was also generally true that the length of period for Mathematics was greater in boys' schools than in girls'. Double periods have been found very valuable in some girls' schools which have tried them.

Mathematics are often begun too late. Girls of 11 shew keen interest in the work, and the more advanced arithmetic which is often done at this stage might well be postponed. The familiarity with mathematical ideas which

an early start gives, lessens the pressure for the years 13 and 14, when girls seem less capable of making effort though they do not necessarily lose interest.

Entrants to secondary schools often shew distaste for and a very poor knowledge of Arithmetic, and this is especially the case when they come from private schools; the variation between school and school shewn in the entrance scholarship and free-place examinations indicates that the difference depends on the kind of teaching rather than on lack of ability in the children. Girls sent on from secondary schools to training colleges, though they shew promise of becoming good teachers of some subjects, are not necessarily capable of teaching Arithmetic, and when this is the case they should not be required to do so. Though the Committee feels that it has little knowledge of the inside working of elementary schools, it would like to draw attention to the fact that some elementary schools which have tried specialist teaching have found the plan a stimulus to both children and teachers. The drawbacks to extensive specialist teaching for very young children are obvious, but would be lessened if subjects were grouped, *e.g.* one teacher might have English, History and Reading, and another Arithmetic, Geography and Nature Study.

The Committee welcomes the recommendation of the Departmental Committee on the Training of Teachers abolishing the student-teacher year, and urging that the minimum qualification for entrance to a training college should be one year's work beyond the stage of a first school examination, and that pupil-teacher centres should be discontinued in urban areas at once and in rural areas as soon as secondary school accommodation is available.

The Committee feels very strongly that it is vital that the early work in Mathematics should be in the hands of those who are both good teachers and good mathematicians. Unfortunately, the demand for well-qualified mathematical mistresses exceeds the supply. Only $33\frac{1}{3}\%$ of the women teachers of Mathematics in the representative schools in which enquiry was made had an honours degree in Mathematics, and only 50% had any kind of training, and these were chiefly those less qualified mathematically. The number of students taking Mathematical Honours at the Universities is increasing very slowly, partly owing, at least at Cambridge, to lack of accommodation for women. The main sources of supply of mathematical specialists at present appear to be Cambridge and London, and many girls who might read for Mathematical Honours do not have the opportunity for want of scholarships. Statistics shew that scholarships for girls tend to go to those in schools where there are highly qualified mathematical teachers, often to mixed schools in which boys and girls have equal opportunities.

The returns made shew that comparatively few girls take post-matriculation work in Mathematics, and of those some are doing one year's work only—hence the inadequacy of supply of competent mathematical teachers. If girls who shewed little capacity for the subject after three years were allowed to drop it, extra teaching might be given to those who shew natural aptitude.

The conclusion reached is that the present standard of attainment in girls' mathematical work can be improved without making demand for more time than is now given. The first need is to increase the supply of well-qualified teachers of the subject. Girls who shew mathematical ability should be encouraged to do post-matriculation work, which should include both Pure and Applied Mathematics and Physics, and should train for teaching under an expert mathematical teacher. The second need is to use the time already allotted to better advantage.

(1) The foundations of the mathematical work should be laid by those who are both good teachers and good mathematicians.

(2) One teacher should take all the work of a mathematical set, and be able to take any branch in any lesson.

(3) In school certificate and later work, double periods will be found to be economical, since they give more possibility of obtaining a good grip of new work and more scope for discussion of interesting points.

(4) Girls shewing ability at any stage should be encouraged to work ahead independently.

While it is probable that girls will in general not achieve as much in Mathematics as boys, it should be possible both to remove the strain experienced at present by far too many girls in efforts to reach the "credit" standard of the School Certificate examination, and to disprove the well-worn but unsound statement that "Girls can't do Mathematics."

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343. Charles Lamb to Thomas Manning. August 24, 1800.—. . . Have you a copy of your Algebra to give away? I do not ask it for myself; I have too much reverence for the Black Arts ever to approach thy circle, illustrious Trismegist! But that worthy man and excellent poet, George Dyer, made me a visit yesternight, on purpose to borrow one, supposing, rationally enough I must say, that you had made me a present of one before this; the omission of which I take to have proceeded only from negligence, but it is a fault. I could lend him no assistance. You must know he is just now diverted from the pursuit of Bell Letters by a paradox, which he has heard his friend Frend (that learned mathematician) maintain, that the negative quantities of mathematicians were *merae nugae*, things scarcely *in rerum naturâ*, and smacking too much of mystery for gentlemen of Mr. Frend's clear Unitarian capacity. However, the dispute once set a-going has seized violently on George's pericranick; and it is necessary for his health that he should speedily come to a resolution of his doubts. He goes about teasing his friends with his new mathematics; he even frantically talks of purchasing Manning's Algebra, which shows him far gone, for, to my knowledge, he has not been master of seven shillings a good time. George's pockets and . . .'s brains are two things in nature which do not abhor a vacuum. . . . Now, if you could step in, in this trembling suspense of his reason, and he should find on Saturday morning, lying for him at the Porter's Lodge, Clifford's Inn,—his safest address—Manning's Algebra, with a neat manuscriptum in the blank leaf, running thus, From the Author! it might save his wits and restore the unhappy author to those studies of poetry and criticism, which are at present suspended, to the infinite regret of the whole literary world.

N.B.—Dirty backs, smeared leaves, and dogs' ears, will be rather a recommendation than otherwise. *N.B.*—He must have the book as soon as possible, or nothing can withhold him from madly purchasing the book on tick. . . . [T. Manning, of Caius, *An Introduction to Arithmetic and Algebra*, 1796, 1798.]—*Lamb's Works*, ed. E. V. Lucas, vi. p. 181. [*per* Mr. F. Puryer White.]

344. Gelasimus . . . cultivated the mathematical sciences with indefatigable diligence, discovered many useful theorems, discussed with great acumen the resistance of fluids, and, though his priority was not generally acknowledged, was the first who fully explained all the properties of the catenarian curves. . . . *The Rambler*, p. 228, No. 179, Vol. 6 (Murphy's edition).

345. Aug. 8, 1666. Discoursed with Mr. Hooke about the nature of sounds, and he did make me understand the nature of musicall sounds made by strings, mighty prettily; and told me that having come to a certain number of vibrations proper to make any tone, he is able to tell how many strokes a fly makes with her wings, (those flies that hum in their flying) by the note that it answers to in musique, during their flying. That I suppose is a little too much refined; but his discourse in general of sound was mighty fine.—Pepys's *Diary*. [*per* the Rev. J. J. Milne.]

A THEOREM CONCERNING POSITIVE INTEGERS THAT ARE PRIME TO EACH OTHER,

WITH APPLICATIONS TO PURE RECURRING DECIMALS.

By C. Tweedie.

The following lesson in Theoretical Arithmetic was originally intended as an Appendix to a Higher Arithmetic, and is now published in the hope that it may be of interest to the more select band of young scholars who may wish to know more of the subject than its application to the calculations of Stocks and Shares, *et hoc omne genus*.

No apology is required for the introduction of algebraic symbols at this stage for such pupils; but the restriction to positive integers only is understood throughout. For the corresponding theorem, applicable to Algebraical Integral Functions of one variable, see Chrystal's *Algebra*, Vol. I.

Theorem. *If a and b are two integers prime to each other, pairs of integers, A and B, may be found such that*

$$Aa - Bb = 1.$$

This is a direct consequence of the long rule for the G.C.M. (or H.C.F.) of two numbers. Every remainder in the process for finding the G.C.M. of a and b may be so represented.

In particular, when a and b are prime to each other, so that the last remainder is unity, A and B may be found such that either

$$Aa - Bb = 1 \quad \text{or} \quad Aa - Bb = -1.$$

E.g. if $a=16$, $b=35$, the successive remainders are 3, 1, where $3 = b - 2a$; $1 = 16 - 5 \times 3 = a - 5(b - 2a) = 11a - 5b$.

If A and B satisfy the relation

$$Aa - Bb = 1,$$

so do

$$A' = A \pm nb; \quad B' = B \pm na,$$

where n is any integer.

For $$A'a - B'b = (A \pm nb)a - (B \pm na)b = Aa - Bb = 1.$$

Also, if the relation found be

$$-Aa + Bb = 1,$$

take n so large that $A - nb$ is a negative number

$$= -A', \text{ say,}$$

or

$$A' - nb = -A;$$

$$\therefore (A' - nb)a + Bb = 1,$$

i.e.

$$A'a - B'b = 1 \quad \text{if } B' = na - B$$

(since $A'a > 1$, B' must be positive).

We may, therefore, assume $Aa - Bb = 1$ .....(I.)
as a characteristic relation connecting a and b, when a and b are prime to each other.

Cor. Clearly A is prime to B and to b, and so is a.

Also,

$$\text{if } nb > A, \quad \text{then } na > B;$$

$$\text{if } nb < A, \quad \text{then } na < B.$$

Theorem. *If a is prime to b and ap/b is an integer, then must b be a factor of p.*

For we may find A and B such that $Aa - Bb = 1$;

$$\therefore Aap - Bbp = p;$$

$$\therefore A \,.\, ap/b - Bp = p/b;$$

i.e. $\frac{p}{b}$ is the difference of two integers, and is therefore an integer.

COR. If $Aa - Bb = 1$, and also $A'a - B'b = 1$, then

$$A - A' = kb \quad \text{and} \quad B - B' = ka,$$

where k is an integer. We suppose $A > A'$.

For $$Aa - A'a = Bb - B'b;$$

$$\therefore (A - A')a/b = B - B' = \text{some integer.}$$

But b is prime to a, and must therefore divide $A - A'$; or $A - A' = kb$; and $B - B' = ka$.

PURE RECURRING DECIMALS.

THEOREM. *If a/n is a proper fraction at its lowest terms, and if n is prime to 10 (the base of notation), then the reduction of a/n to the decimal form gives rise to a pure recurring decimal. In particular, $1/n$ gives rise to a pure recurring decimal.*

DEM. The process of reduction introduces as a factor the base 10, or a power of 10. Hence all the remainders in the process are prime to n, and, being all less than n, they must ultimately repeat.

Suppose $1/n = \cdot pq \ldots r\dot{d}_1d_2 \ldots \dot{d}_t$, where $p, q, \ldots r$ denote the non-recurring figures of the resulting decimal, and $d_1, d_2, \ldots d_t$ represent the recurring figures.

If α is the remainder after r is taken in the decimal, then α is prime to n, and

$$\alpha/n = \dot{d}_1d_2 \ldots \dot{d}_t,$$

in which there are t digits in the recurring cycle.

But, α being prime to n, A and B may be found so that

$$Aa - Bn = 1.$$

Hence $$1/n = A \times a/n - B$$

$$= A \times \dot{d}_1d_2 \ldots \dot{d}_t - B.$$

Now the factor A applied to $\dot{d}_1d_2 \ldots \dot{d}_t$ will furnish a decimal part with a cycle of t (or fewer) digits; and B will cancel the integral part, since $1/n$ is a proper fraction.

Hence $1/n$ is also a pure recurring decimal of t (or fewer) digits.

On the other hand, $1/n$ cannot contain fewer than t digits in its cycle, as unrepeated, for

$$a/n = a \times 1/n;$$

and therefore a/n cannot contain more digits in its cycle than $1/n$.

Similarly, if a is any other integer less than n, and prime to n, then

$$a/n = a \times 1/n = a \times \text{a recurring cycle of } t \text{ digits.}$$

Moreover, we can find A' and B' so that

$$1/n = A'a/n - B';$$

$\therefore$ a/n must be a pure recurring decimal of t digits.

Furthermore, if $$a/n = \dot{b}_1b_2 \ldots \dot{b}_t$$

furnishes the remainders $$a_1, a_2, \ldots a_t = a,$$

then $$a_1/n = \dot{b}_2b_3 \ldots b_t\dot{b}_1,$$

$$a_2/n = \dot{b}_3b_4 \ldots b_t\dot{b}_1\dot{b}_2,$$

and so on, with a corresponding cycle of the same remainders.

The different numbers a less than n, and prime to n, thus arrange themselves in separate groups of t. Hence t is a factor in the number which indicates how many integers there are less than n and prime to n.

E.g. If $n = 13$, t is a factor of 12.

Thus $1/13 = \cdot\dot{0}76923\dot{}$,
and the remainders are 10, 9, 12, 3, 4, 1.
Also $2/13 = \cdot\dot{1}5384\dot{6}$,
with remainders 7, 5, 11, 6, 8, 2.

There are thus, as remainders, two cycles of 6 integers, each < 13, and $t = 6$.
If $n = 21$, t is a factor of 12, there being 12 integers less than 21 and prime to it.

Cor. 1. *The number* $(10^t - 1)/n$ *is an integer if* n *is prime to* 10.

For $$1/n = \dot{d}_1 d_2 \ldots \dot{d}_t$$
$$= (d_1 d_2 \ldots d_t)/999 \ldots 9.$$

Hence n is a factor in the denominator, *i.e.* in $10^t - 1$.
In fact, $$(10^t - 1)/n = 10^t \times 1/n - 1/\dot{n}$$
$$= d_1 d_2 \ldots d_t.$$

Cor. 2. Denote by $\phi(n)$ the number of integers less than n and prime to n;
$\therefore$ $10^t - 1$ is a factor of $10^{\phi(n)} - 1$, so that $(10^{\phi(n)} - 1)/n$ is an integer.

Cor. 3. The theory has so far been worked out for the ordinary decimal notation of base 10. But the same theory will be easily seen to be true for any other system of base b.

I.e. If n is prime to the base b, then

$$1/n = \dot{d}_1 d_2 \ldots \dot{d}_t = d_1/b + d_2/b^2 + \ldots + \frac{d_t}{b^t} + \frac{d_1}{b^{t+1}} + \ldots$$
$$= (d_1 d_2 \ldots d_t)/(b^t - 1),$$

so that n is a factor of $b^t - 1$, where t is a factor of $\phi(n)$.

EXAMPLES.

Pure recurring decimals have many interesting properties. They are more pompous in their array than the plainer vulgar fractions, to which may often be traced their elegant evolutions. In their investigation constant attention must be paid to the successive remainders. It will be found that there are always *two* cycles, viz. the cycle of digits in the decimal, and a corresponding cycle of remainders, as will appear presently.

To save space, denote by t the number of digits in the cycle.

If $a + b = n$, call a and b complementary numbers with respect to n.

The following theorem is often used. Its proof is usually given in Arithmetics. The Remainder after dividing the number whose digits are

$$n_1 n_2 n_3 \ldots n_{\alpha-k} a_1 a_2 \ldots a_\alpha \ldots l_1 l_2 \ldots l_\alpha$$

by the number 9999..., where there are α nines,

$$= \text{Remainder for } (n_1 n_2 \ldots n_{\alpha-k} + a_1 a_2 a_\alpha + \ldots l_1 l_2 \ldots l_\alpha)$$

when divided by the same number.

E.g. Remainder after division of 68,543,672 by 999
$$= \text{Remainder for } (68 + 543 + 673)/999$$
$$= \text{Remainder for } 1283/999 = 284.$$

Ex. 1. Consider the fractions
$\frac{1}{7} = \cdot\dot{1}42857\dot{}$,
with remainders 326451.
$\frac{3}{7} = \cdot\dot{2}8571\dot{4}$,
with remainders 645132.
Etc.

We note that the digits in the different decimals are the same in all, and in the same cyclical order. The same is true for the Remainders. There is, in

fact, only one pair of cycles, one for the decimals, and one for the remainders, and they may be denoted by the notation :

$$\cdot\dot{1}42857\dot{7}$$
$$326451,$$

or by

$$\cdot\dot{4}2857\dot{1}$$
$$264513, \text{ etc.}$$

(i) In each cycle the digits of the decimal are complementary in pairs with respect to 9, and the remainders are also complementary in pairs, but with respect to 7.

Thus
$$1+8=4+5=2+7=9,$$
$$3+4=2+5=6+1=7.$$

From the first three numbers in either cycle, the remaining three can be at once written down.

(ii) The sums $28+57+14$, $85+71+42$ are divisible by 99.
Of the remainders, $3+6+5$, $2+4+1$ are divisible by 7.

(iii) The sum $1+4+2+8+5+7$ is divisible by 9 ; the sum of the remainders is also divisible by 7.

(iv) Multiplication of a decimal by any number $7n+a$ $(a<7)$ reproduces a decimal with the same cycle *plus* a multiple of n.
The proof will appear in the next example.

Ex. 2. Consider the fractions

$$1/13 = \cdot\dot{0}\ 7\ \ 6\ 9\ 2\ \dot{3} \quad \text{.....(i)}$$
$$10\ 9\ 12\ 3\ 4\ 1.$$
$$2/13 = \cdot\dot{1}\ 5\ \ 3\ 8\ 4\ \dot{6} \quad \text{.....(ii)}$$
$$7\ 5\ 11\ 6\ 8\ 2.$$

Here, though $\phi(n)=12$, t is $=6$.
The 12 possible remainders occur in (i) and (ii), and there can be no new cycles.

Thus
$$3/13 = \cdot 2\ 3\ \ 0\ 7\ \ 6\ 9$$
$$4\ 1\ 10\ 9\ 12\ 3.$$

Again, (i) and (ii) teach us that

$$10^3 = 7\times 13+9\ ;\ 10^3 = 76\times 13+12\ ;\ \ldots\ 10^7 = 769230\times 13+10, \text{ etc. ;}$$
$$5\times 10^3 = 384\times 13+8\ ;\ 5\times 10^4 = 3846\times 13+2\ ;\ \text{etc.,}$$

as found by noting the remainders in the cycles.

Now
$$1/13 = (076923)/(999{,}999).$$

There are 6 nines in the denominator on the right, and it is therefore divisible by 999 ; 99 ; 9 ; none of which contains the prime number 13 (for which 999,999 is the first of the type, as $t=6$).

Hence $076+923$ is divisible by 999, and corresponding digits are complementary in pairs, as the quotient is necessarily unity.

Similarly, 153846 is divisible by 999, and corresponding digits 1, 8 ; 5, 4 ; 3, 6 ; are complementary with respect to 9. From divisibility by 99, it follows that the sums

$$7+69+23\ ;\ \ 76+92+30\ ;$$
$$15+38+46\ ;\ \ 53+84+61\ ;$$

are each divisible by 99.

Finally, $7+6+9+2+3$; $1+5+3+8+4+6$ are each divisible by 9.

Again, the quotient of 999,999 by 999 is 1001 ;

$$\therefore\ (10^3+1)/13 \text{ is an integer,}$$
$$10^3/13 = 76\tfrac{2}{13} \quad \text{and} \quad \tfrac{1}{13}=\tfrac{1}{13}.$$

Likewise 13 is a factor in

$$10010, \quad i.e. \text{ in } 10^4+10.$$

Thus $10^4/13+10/13$ and also $10^5/10+10^2/13$ are integral.

Hence remainders in (i) corresponding to digits, complementary with respect to 9, are complementary with respect to 13.

The same law is observed in (ii).

Division of 999,999 by 99 gives 10101 divisible by 13, and so are 20102, 30103, etc.

Thus $(10^4+10^2+1)\div 13$ is integral,

or $2(10^4+10^2+1)\div 13$ is integral, etc.

Hence $1+9+3;\quad 10+12+4;$

$7+11+8;\quad 5+6+2;$

are all divisible by 13: and so are the sums of all the remainders in (i) or (ii). We may note that in (i) 1 and 12 are self-conjugate with respect to 13, and the other four are remainders conjugate in pairs. In (ii) there are three conjugate pairs. But this property is not true in general.

With respect to multiplication by a number $13n+\alpha$, where $\alpha<13$:—

If α is any of the remainders in (i), there results from (i) a number whose decimal has the same cycle as in (i), and from (ii) a cycle of (ii).

If α is any remainder of (ii), then the cycle (i) is transformed into (ii), and (ii) is transformed into (i).

Ex. 3. If t for $1/n$ as a decimal is even $=2\lambda$, and if n is a prime number, then the cycles of digits show the complementary property.

For, if $t=2\lambda$, $\quad (10^{2\lambda}-1)/n=(10^\lambda-1)(10^\lambda+1)/n=$ an integer.

Now, by hypothesis, $(10^\lambda-1)/n$ is not integral;

$$\therefore\ 10^\lambda+1 \text{ is divisible by } n=Kn, \text{ say,}$$

so that $$10^\lambda=(K-1)n+n-1.$$

Hence in the cycle of remainders for $1/n$, the remainders λ and 2λ in order are complementary. The proof may then be completed.

Ex. 4. The decimals given by $\frac{1}{41}$, $\frac{2}{41}$, etc., have $t=5$, so that there are 8 pairs of cycles of digits and remainders. We find the complete set from

$$\begin{array}{rl} 1/41= & \cdot\dot{0}\ 2\ 4\ 3\ \dot{9} \\ & 10\ 18\ 16\ 37\ 1 \\ 2/41= & \cdot\dot{0}\ 4\ 8\ 7\ \dot{8} \\ & 20\ 36\ 32\ 33\ 2 \\ 3/41= & \cdot\dot{0}\ 7\ 3\ 1\ \dot{7} \\ & 30\ 13\ 7\ 29\ 3 \\ 4/41= & \cdot\dot{0}\ 9\ 7\ 5\ \dot{6} \\ & 40\ 31\ 23\ 25\ 4 \\ 5/41= & \cdot\dot{1}\ 2\ 1\ 9\ \dot{5} \\ & 9\ 8\ 39\ 21\ 5 \\ 6/41= & \cdot\dot{1}\ 4\ 6\ 3\ \dot{4} \\ & 19\ 26\ 14\ 17\ 6 \\ 11/41= & \cdot\dot{2}\ 6\ 8\ 2\ \dot{9} \\ & 28\ 34\ 12\ 38\ 11 \\ 15/41= & \cdot\dot{3}\ 6\ 5\ 8\ \dot{5} \\ & 27\ 24\ 35\ 22\ 15 \end{array}$$

Denote 11 by e, and 15 by f.

Indicate the cycle pairs as

$$1,\ 2,\ 3,\ 4,\ 5,\ 6,\ e,\ f.$$

Note that 1 and 4; 2 and 5; 3 and e; 6 and f; furnish complementary

cycles. In 1 or in 4 there is a self-conjugate remainder (for 41), and the other four remainders are conjugate in pairs: 2 and 5; 3 and 6; e and f; furnish conjugate remainders in *inverse* order.

Multiplication by $41n+a$.

If a is any remainder of a given cycle, it always transforms a second cycle into a perfectly definite third cycle. The following table is the result:

MULTIPLICATION BY A REMAINDER

			1	2	3	4	5	6	e	f
From	1	gives	1	2	3	4	5	6	e	f
„	2	„	2	4	6	5	1	e	f	3
„	3	„	3	6	5	e	f	1	2	4
„	4	„	4	5	e	1	2	f	3	6
„	5	„	5	1	f	2	4	3	6	e
„	6	„	6	e	1	f	3	2	4	5
„	e	„	e	f	2	3	6	4	5	1
„	f	„	f	3	4	6	e	5	1	2

The square array in the table has some interesting properties. No row nor column contains two identical numbers. This alone facilitates its construction. Corresponding rows and columns are identical. It is symmetrical with respect to one of its diagonals. More remarkable still, if we apply to any row in it the same multiplication by 1, 2, 3, etc., there results simply a square array with the same rows as before, but in a different order.

Space forbids the explanation of these laws, but the pupil may try his hand on 1/11 and 1/37, etc.

Ex. 5. Show that a/n and $(n-a)/n$ furnish two decimal fractions, whose digits are complementary with respect to 9, when a is prime to n, while corresponding remainders are complementary with respect to n (*e.g.* 5/21 and 16/21).

If, when n is a prime number, t is composite $=mK$, and if

$$a/n = \cdot\dot{d}_1 d_2 d_3 \ldots d_m d_{m+1} \ldots \dot{d}_{mK},$$

$$r_1 r_2 r_3 \ldots r_m \ldots\ldots\ldots r_{mK},$$

prove

(i) $d_1 d_2 \ldots d_m + d_{m+1} \ldots d_{2m} + \ldots$, etc., is divisible by $999\ldots$, where there are m nines.

(ii) $r_1 + r_{m+1} + r_{2m+1} +$, etc., is divisible by n.

The same theorem is true when the base of notation is not 10 but t, say, but substitute $t' = t-1$ for 9 throughout.

Ex. 6. Remainders with respect to the members of a Geometric Progression $K + Ka + Ka^2 + \ldots + Ka^m$, where $K < n$, and K and a both prime to n. Examination will show that this problem has been pretty fully discussed when a is 10.

For any other integer a, select a as the base of notation, when the laws of variation or repetition of remainders may readily be discussed. Cycles of remainders will arise depending on the value of t and on K.

Ex. 7. Construct tables for the last two digits in all powers of numbers from 1 to 100. If A is any number not divisible by 5, A^{20} has either 76 or 01 for the last two digits, and either $A^{22} - A^2$ or $A^{21} - A$ is divisible by 100.

C. TWEEDIE.

MATHEMATICAL NOTES.

816. [K¹. 10.] *A Holiday Problem.*

If the vertex A of a triangle is real, but B and C are conjugate complex points on a given line, how much survives of the figure ordinarily associated with a triangle, and what real constructions can be carried out? To make the questions more precise, B and C may be supposed given as the intersections of the base with the circumcircle, which is therefore a given circle through A which does not cut the given line.

Since complex elements must occur in pairs, any point or line which for an ordinary triangle is unique must remain. For example, there is an orthocentre, and any point of the circumcircle retains its Simson line. But the incentre is not in this sense a unique point, for analytically it is one of a set of four contact-centres.

My object now is not to discuss the problem, but to suggest that some readers may find entertainment in worrying at it. The matter is not wholly frivolous. For example, it is well known that if a variable tangent to a conic cuts the tangents from a fixed point A outside the conic in P and Q, the locus of the orthocentre of APQ is a hyperbola. If A is inside the conic, the locus survives in the form of an ellipse. What is the simplest description, in real terms, of the point of which the ellipse is the locus? E. H. N.

817. [O¹. δ. a.] *Two Propositions on a Moving Lamina.*

1. If a line AB in the lamina becomes $A'B'$, to prove that IP is perpendicular to MN, where I is the centre of rotation, P the point of intersection of AB and $A'B'$; and M, N are the middle points of AA' and BB' respectively.

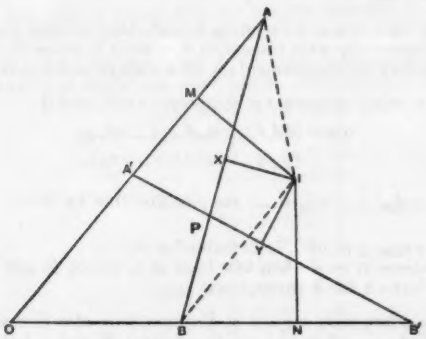


FIG. 1.

In Fig. 1, I is the point of intersection of perpendiculars to AA' and BB' through M and N. Draw IX and IY perp. to AB and $A'B'$. Let AA' and $B'B$ meet in O. Join IA, IB.

Then the triangles IAB, $IA'B'$ are congruent; $\therefore IX = IY$.

Also $A\hat{I}A' = B\hat{I}B'$ or $A\hat{I}M = B\hat{I}N$.

Hence $A\hat{I}B = M\hat{I}N = 180° - \hat{O}$.

And similarly $A'\hat{I}B' = 180° - \hat{O}$.

Hence the circumcircles of the triangles AOB, $A'OB'$ meet in I, and $MXYN$ is the common pedal line of the pt. I. But IP is obviously $\perp$ to XY, and therefore to MN.

Cor. The point of contact of any line in the lamina with its envelope is obtained by drawing to it a perp. from the instantaneous centre.

2. To find the radius of curvature of the path of a point in the lamina by a dynamical method.

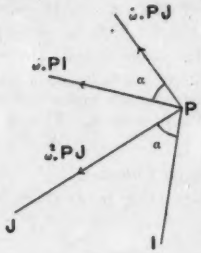


FIG. 2.

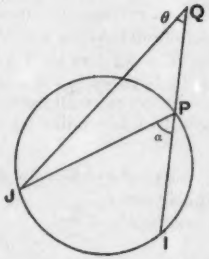


FIG. 3.

If I is the point of no velocity and J the point of no acceleration, and the angular velocity and angular acceleration are given, the velocity and acceleration of a point P are as indicated in Fig. 2. For a point of inflexion the resultant acceleration must coincide in direction with that of the velocity. Hence the angle $IPJ = \tan^{-1}(\omega^2/\dot{\omega})$, and is given; so that the locus of such points is a circle through I and J.

For any other point Q the normal acceleration along QI is $\omega^2 . QI^2/\rho$, and this is equal to $QJ\ (\omega^2 \cos\theta - \dot{\omega} \sin\theta)$

$$= \omega^2 QJ\ (\cos\theta - \cot\alpha \sin\theta)$$
$$= \omega^2 QJ \sin(\alpha - \theta)/\sin\alpha$$
$$= \omega^2 . QP \text{ from the triangle } QIP.$$

Hence $\rho = QI^2/QP$. N. M. GIBBINS.

818. [J.2.b.] *Tchebycheff's Theorem.*

Writers on probability and statistics make some use of an algebraical theorem due to Tchebycheff. The usual proof is rather tedious, but, as I have already pointed out elsewhere,* the reasoning can be expressed quite simply.

(i) Let $X_1, X_2, \ldots X_N$ be N quantities of the same kind. Let their average value be A, and let the sum of the squares of their deviations from the average be ND^2. Then the proportion of cases for which $X - A$ is numerically greater than λD, where λ is any constant >1, cannot exceed $1/\lambda^2$.

Proof. If the proportion of these cases did exceed $1/\lambda^2$, then the contribution made by them to ND^2 would exceed $N/\lambda^2 . (\lambda D)^2$, *i.e.* would exceed ND^2; which is impossible.

(ii) We can then limit the theorem to cases in which the X's are the values of a variable quantity X. Changing the statement slightly, it becomes:

In the frequency-distribution of a quantity X, let the mean value be A, and the dispersion (standard deviation) be D. Then the relative frequency of cases for which $|X - A|$ does not exceed λD, where λ is any constant >1, is not less than $1 - 1/\lambda^2$.

* *Journal of Institute of Actuaries*, liii. (1922), 82.

By taking X to be the sum of several independent variables $Y_1, Y_2, \ldots Y_n$, we get Tchebycheff's theorem. But this can be left till it is required. The more general statement given above is simpler to work from.

(iii) Take, for example, the proof—which, I suppose, is due to Tchebycheff himself—of Bernoulli's theorem.

Let the probability of an event, at each of n trials, be p; let the actual number of times of its happening in the n trials be m; and let us take $X \equiv m/n$, the proportionate number of times of happening. Then, for the frequency-distribution of X, n being constant, we know that $A=p$, $D=\sqrt{pq/n}$, where $q=1-p$. It follows that the probability that m/n will lie between $p-\lambda\sqrt{pq/n}$ and $p+\lambda\sqrt{pq/n}$ is not less than $1-1/\lambda^2$. In other words, the probability that m/n will lie between $p-\alpha$ and $p+\alpha$, where $\alpha \equiv \lambda\sqrt{pq/n}$, is not less than $1-pq/(n\alpha^2)$. However small we take α to be, we can make this probability as nearly equal to 1 as we like by taking n large enough. This is Bernoulli's theorem.

W. F. SHEPPARD.

819. [V. 1. a. α.] *Suggestion for first lessons on Calculus.*

1. Definition of differential coefficient and proof that (*a*) $D(c)=0$, (*b*) $D(x)=1$, and (*c*) $(uv)'=wv'+u'v$.

2. From 1 (*c*),
$$\frac{(uv)'}{uv}=\frac{u'}{u}+\frac{v'}{v}.$$

Extend to 3 or more functions;
$$\therefore \frac{(u_1u_2\ldots u_n)'}{u_1u_2\ldots u_n}=\Sigma\frac{u_r'}{u_r}.$$

Make
$$u_1=u_2=\ldots=u_n\equiv u;$$
$$\therefore \frac{(u^n)'}{u^n}=n\frac{u'}{u} \quad \text{or} \quad (u^n)'=nu^{n-1}u',$$

where n is +ve int.*

3. Let $n=p/q$ and $v=w^{\frac{p}{q}}$, where p and q are +ve ints.;
$$\therefore v^q=w^p;$$
$$\therefore qv^{q-1}v'=pw^{p-1}w';$$
$$\therefore v'=\frac{p}{q}\frac{w^{p-1}}{v^{q-1}}w'=\frac{p}{q}w^{\frac{p}{q}-1}w'=nw^{n-1}w'.$$

Let $n=-m$ and $w=u^{-m}$, where m is +ve;
$$\therefore wu^m=1.$$

By 1 (*c*) and 2,
$$w'u^m+wmu^{m-1}u'=0;$$
$$\therefore w'=-m\frac{w}{u}u'=-mu^{-m-1}u'=nu^{n-1}u'.$$

Hence always
$$(u^n)'=nu^{n-1}u',$$
and in particular
$$(x^n)'=nx^{n-1}.$$

4. Drill in differentiating algebraic functions.

5. Applications.

The advantages claimed is that the use of the Binomial Theorem (any index), and the theorem on the differentiation of a function of a function, are avoided at this early stage.

N. M. GIBBINS.

*Shewn to me by Rev. E. M. Radford.

820. [K¹. 2. d.] *Note on Feuerbach's Theorem.*

Let A, B, C denote the vertices of a euclidean plane triangle, A_1, B_1, C_1 the midpoints of the sides BC, CA, AB respectively, G the centroid, I the incentre, and N the nine-point centre of ABC, F the point of contact of the incircle and the nine-point circle, f their common tangent at F. Let the line GI meet the lines B_1C_1, C_1A_1, A_1B_1 at P, Q, R respectively, and let P', Q', R' respectively denote the harmonic conjugates of P, Q, R with respect to B_1, C_1, C_1 and A_1, A_1 and B_1 respectively. Then the circles, p, q, r, on PP', QQ', RR' respectively as diameters, touch at F, their common tangent being the line IN, and their line of centres the Feuerbach tangent f.

For it is known that the lines FP, FQ, FR (Mannheim's lines) are respectively perpendicular to the lines FP', FQ', FR' (Hamilton's lines); thus the circles p, q, r meet at F. Also P, P'; Q, Q'; R, R' are the pairs of opposite vertices of a complete quadrilateral, of which $A_1B_1C_1$ is the diagonal triangle; the nine-point circle of ABC, which contains F, is thus the circumcircle of the diagonal triangle of the quadrilateral, and is therefore orthogonal to each of the three circles p, q, r. The theorem follows immediately.

The Feuerbach tangent f is the medial line of the quadrilateral, and the line IN its orthocentric line. The quadrilateral is conjugate to every circle which touches the incircle at F.

(For references to the lines of Hamilton and Mannheim, see *Mathesis*, xxxvi. (1922), p. 51.) J. P. GABBATT.

821. [V.] *Note on Mr. Langley's article, Gazette*, xii. p. 468.

The original accounts of the expenses of the Northamptonshire quota of infantry in the Armada year are in the Public Records Office, State Papers Domestic, ccxiv. 32 and 33. In both documents the year is written as 1588; and in the former "150 men" occurs three times, "one hund' and fyftie" once, "fyfteen shillinges" once, and "50 men" twice. Otherwise every figure is in Roman character.

But there are copies, not quite complete, in a "muster-book" which was evidently kept for private reference. Here "xs." occurs five times, "xv.s" once, and "vj cartes" once; and occasionally "fyve daies," "fouer score poundes of pouldear," and other phrases, are written. Otherwise all the figures are Hindu-Arabic.

Now at that date would it have been the ordinary custom for a clerk, the chief captain's secretary, to write out a bill for private reference in Hindu-Arabic, but the official bill, which had to be sent up to government, in Roman?

The point at issue is this: if it was the common custom, then the copy was entered in the muster-book at once; if not so, then the clerk made a copy at once, but entered in the book a copy of this copy at a later date. In short, if some of your readers who have studied the dates of the Arabic notation would be so good as to give an opinion, it would help myself and others interested to decide whether the entries in the muster-book were contemporary with the facts recorded. JOHN E. MORRIS.

Wokingham.

822. [K¹. 6.] *The sign of $ax+by+c$.*

The point dividing $(x_1y_1)(x_2y_2)$ in the ratio $k:1$ internally has coordinates

$$\frac{x_1+kx_2}{1+k},\ \frac{y_1+ky_2}{1+k},\ \text{and } k \text{ is positive.}$$

If this point lies on $ax+by+c=0$, $(ax_1+by_1+c)+k(ax_2+by_2+c)=0$.

k being positive, ax_1+by_1+c and ax_2+by_2+c must have different signs, or, in other words, $ax+by+c$ changes sign on crossing the line $ax+by+c=0$.

College of Technology, Manchester. H. V. LOWRY.

823. [V. 4.] *Euclid* (IV. 10).

To describe a triangle having each base angle twice the vertical angle.

(*a*) Divide the line PQ in O so that $PQ \cdot QO = PO^2$. [PO is $> QO$ (II. xi).]

With O as centre and OP as radius describe the circle PRT; with Q as centre and with the same radius PO describe circle SRH.

This second circle *must* intersect former circle in some point (R), since its radius is $> QT$ and less than QP.

Join QR and OR.

Proof. Since $OR = OP$,

$$\therefore \hat{ORP} = \hat{OPR};$$

and since $PQ \cdot QO = OP^2 = OR^2$,

$\therefore$ QR is tangential to the circle POR;

$$\therefore \hat{ORQ} = \hat{OPR};$$

$$\therefore P\hat{R}Q = 2\,\hat{OPR};$$

and since lines OP, OR, QR are all equal,

$$\therefore P\hat{Q}R = Q\hat{O}R = 2\,\hat{OPR}. \quad \text{(Q.E.D.)}$$

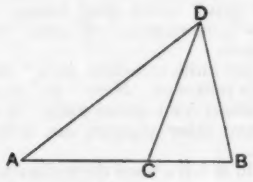
J. H. CULLEN.

(*b*) *Constr.* Draw any st. line AB.

Divide it in extreme and mean ratio at C, so that $AB : AC :: AC : CB$.

With centres C, B, and radius AC, draw two arcs intersecting at D.

Join CD, DB, AD.



Prove. ADB or CDB the Δ required.

Proof. $AB : AC :: AC : CB$;

$$AB : BD :: BD : CB.$$

Also $\angle B$ is common to Δs ABD, CBD;

$\therefore$ these Δs are similar.

But CDB is isosceles;

$\therefore$ $AD = AB$, and $\angle ADB = \angle ABD$.

Again, $AB : AC :: AC : CB$;

$\therefore$ $AD : DB :: AC : CB$;

$\therefore$ CD bisects $\angle ADB$.

$\therefore$ ΔADB (consequently DCB) has its base angles double of $A\hat{D}C$, *i.e.* of $\angle A$.

High School, Newcastle, N.S.W. A. L. NAIRN.

824. [B. 1.] *A Curious Determinant.*

Let ordinates $y_0, y_1, y_2 \dots y_{2n}$ be drawn at equal intervals, width h, and the middle ordinate be taken as y-axis, and let a curve $y=a_0+a_1x+\dots+a_kx^k$ be drawn through the ends of these ordinates.

The area under the curve between y_0 and y_{2n} is:

$$\int_{-nh}^{+nh}(a_0+a_1x+\dots+a_kx^k)dx=a_0(2nh)+a_2\left(\frac{2n^3h^3}{3}\right)+\dots \quad \dots\dots\dots\dots(1)$$

Let this area be also given by:

$$B_0y_0+B_1y_1+\dots+B_{2n}y_{2n}. \quad \dots\dots\dots\dots(2)$$

Substituting for the y's, this becomes

$$\begin{aligned} &a_0(B_0+B_1+\dots+B_{2n})\\ &\quad +a_1\{-nB_0-(n-1)B_1-(n-2)B_2-\dots+(n-1)B_{2n-1}+nB_{2n}\}h\\ &\quad +a_2\{n^2B_0+(n-1)^2B_1+\dots+(n-1)^2B_{2n-1}+n^2B_{2n}\}h^2\\ &\quad +\dots\dots\dots\dots \quad \dots\dots\dots\dots(3)\end{aligned}$$

Equating (1) and (3),

$$\begin{aligned} &B_0+B_1+B_2+\dots+B_{2n}=2nh,\\ &-nB_0-(n-1)B_1-\dots+(n-1)B_{2n-1}+nB_{2n}=0,\\ &n^2B_0+(n-1)^2B_1+\dots+(n-1)^2B_{2n-1}+n^2B_{2n}=\frac{2n^3h}{3},\end{aligned}$$

etc.

The first $2n+1$ of these equations determine $B_0, B_1, \dots B_{2n}$, and consequently using these values of the B's the area of any curve of degree $2n$ is given correctly by (2).

But these formulae have the peculiarity that the even ones can be solved for $B_{2n}-B_0$, $B_{2n-1}-B_1$, etc., giving $B_{2n}=B_0$, $B_{2n-1}=B_1$, ...

It follows that all the even equations are satisfied regardless of the actual values of the B's, and that (2) not only gives the correct area for a curve of degree $2n$, but also for a curve of any higher degree, whose terms above the $2n^{\text{th}}$ degree are all of odd degree.

Now the area could have been found by taking y_0 to be the y-axis.

If the equation to the curve is then

$$y=b_0+b_1x+\dots+b_kx^k,$$

the area is $$b_0(2nh)+b_1\frac{(2nh)^2}{2}+b_2\frac{(2nh)^3}{3}+\dots+b_k\frac{(2nh)^{k+1}}{k+1}.$$

Equating this to $B_0y_0+B_1y_1+\dots+B_{2n}y_{2n}$ as before,

$$\begin{aligned} &B_0+B_1+B_2+\dots+B_{2n}=2nh. \quad \dots\dots\dots\dots(4)\\ &B_1+2B_2+3B_3+\dots+2nB_{2n}=\frac{2^2n^2h}{2},\\ &B_1+2^2B_2+3^2B_3+\dots+(2n)^2B_{2n}=\frac{2^3n^3h}{3},\end{aligned}$$

....

The 1st $2n+1$ of these equations must give the same values of the B's as before, and we have shown that they must also satisfy

$$B_1+2^sB_2+3^sB_3+\dots+(2n)^sB_{2n}=\frac{2^{s+1}n^{s+1}h}{s+1} \quad \dots\dots\dots\dots(5)$$

for any odd value of s.

Eliminating $B_1, B_2, \ldots B_{2n}$ from the $2n$ equations of (4), after the first, and (5), we have

$$\begin{vmatrix} 1, & 2, & 3, \ldots\ldots & 2n, & \frac{2^2n^2}{2} \\ 1^2, & 2^2, & 3^2, \ldots\ldots & (2n)^2, & \frac{2^3n^3}{3} \\ \ldots\ldots & & & & \ldots\ldots \\ 1^{2n}, & 2^{2n}, & 3^{2n}, \ldots & (2n)^{2n}, & \frac{2^{2n+1}n^{2n+1}}{2n+1} \\ 1^s, & 2^s, & 3^s, \ldots\ldots & (2n)^s, & \frac{2^{s+1}n^{s+1}}{s+1} \end{vmatrix} = 0$$

for all odd values of s.

It is an interesting problem to obtain a direct proof of this result.

College of Technology, Manchester. H. V. LOWRY.

825. [L¹. 17.] *A Property connected with the "Circle of Zero Velocity" for the Ellipse.*

A circle with centre at a focus, S, of an ellipse and radius equal to the major axis is called the circle of zero velocity for the ellipse.

Theorem. Let C be any point on this circle, and let CS produced cut the ellipse at P. Then if the tangent at P cuts the tangents from C to the ellipse at A and B, $CS^2 = SA \,.\, SB$, and the line joining C to the second focus, S', is a median of the triangle ABC.

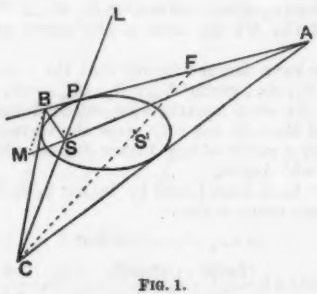


FIG. 1.

Draw the circle ABC cutting CS produced at L and AS produced at M. Now it is easy to show, by using the properties of tangents to a conic from an external point, that CSL bisects the angle ASB.

Again, $CS = 2\rho_1 = \dfrac{AS \,.\, SB}{SL}$; [See *Gazette*, Oct. 1922, p. 169.

$\therefore\ AS \,.\, SB = CS \,.\, SL = AS \,.\, SM$;

$\therefore\ SB = SM$.

Hence $\angle MBS = \frac{1}{2}\angle ASB = \angle BSL$;

$\therefore\ MB$ is par^l to CL,

so arc ML = arc BC;

$\therefore\ \angle MCS = \angle BLS$;

also $\angle MSC = \angle ASL = \angle BSL$

and $SM = SB$.

So $\triangle$s SMC, SBL are congruent and $SC=SL$,

$$\therefore\ CS^2=SA\,.\,SB. \qquad \text{.....(1)}$$

It follows that $\triangle$s BSC, CSA are similar, since $\angle CSB=\angle ASC$;

$$\therefore\ \angle BSL=\angle SBC+\angle SCB=\angle SCA+\angle SCB=\angle ACB.$$

Hence $\triangle$s SBL, CBA are equiangular ($\angle BLS=\angle BAC$);

$$\therefore\ \frac{BL}{LS}=\frac{BA}{AC};$$

$$\therefore\ \frac{BL}{LC}=\frac{FA}{AC},\ \text{where } F \text{ is the midpoint of } AB;$$

also $$\angle BLC=\angle FAC.$$

$\therefore$ $\triangle$s BLC, FAC are equiangular, so $\angle BCS=\angle FCA$;

$$\therefore\ S' \text{ lies on the median, } CF, \text{ of } \triangle ABC. \qquad \text{.....(2)}$$

[It is perhaps worth noting that S is the focus of the parabola touching CA, CB at A and B.]

Again, since S' is the image of L in AB,

$$\angle AFL=\angle AFC,$$

but $$\angle FAL=\angle BCL=\angle FCA;$$

$$\therefore\ \triangle\text{s } LFA,\ AFC \text{ are equiangular;}$$

$$\therefore\ FA^2=FL\,.\,FC.$$

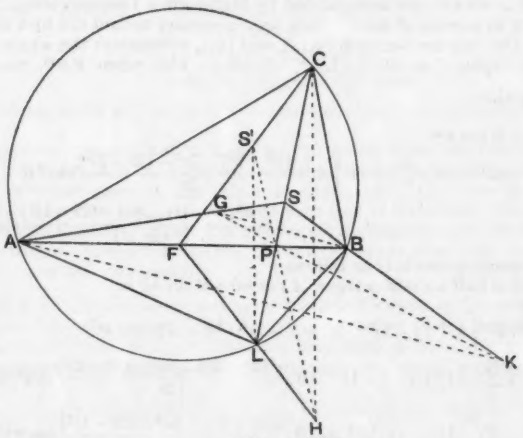


FIG. 2.

So F may be regarded as the focus of an ellipse inscribed in $\triangle ALC$ and touching LC at P.

The second focus, G, will be the image of B in CL, and the major axis will be FA.

There are also two hyperbolas, inscribed respectively in the $\triangle$s ALB, LBC, the foci of the former being S and H, where H is the image of C in AB, and those of the latter F and K, where K is the image of A in CL; the major axis of the former is equal to CS and that of the latter to FA.

These four conics touch in pairs at the point P, the tangents being $AB\,.\,CL$.

May, 1925. E. P. LEWIS.

826. [D.2. b.β.] *Expansion of* $A\cos n\theta + B\sin n\theta$ *in powers of* $\sin\theta$ *or* $\cos\theta$.

Let $y = A\cos n\theta + B\sin n\theta$ and $x = a\cos\theta + b\sin\theta$;

then $\dfrac{d^2y}{d\theta^2} = -n^2y$ and $\dfrac{dy}{d\theta} = \dfrac{dy}{dx}\dfrac{dx}{d\theta} = \dfrac{dy}{dx}(-a\sin\theta + b\cos\theta)$;

$$\therefore \frac{d^2y}{d\theta^2} = \frac{d^2y}{dx^2}(-a\sin\theta + b\cos\theta)^2 + \frac{dy}{dx}(-a\cos\theta - b\sin\theta)$$

$$= \frac{d^2y}{dx^2}(a^2 + b^2 - x^2) - \frac{dy}{dx}x.$$

Equating the two values of $\dfrac{d^2y}{d\theta^2}$ and writing y_r for $\dfrac{d^ry}{dx^r}$,

$$y_2(a^2 + b^2 - x^2) - xy_1 + n^2y = 0.$$

Differentiating k times by Leibniz' Theorem,

$$y_{k+2}(a^2 + b^2 - x^2) + ky_{k+1}(-2x) + \frac{k(k-1)}{1.2}y_k(-2) - xy_{k+1} - k.y_k + n^2y_k = 0;$$

and putting $x = 0$ and $a^2 + b^2 = 1$,

$$[y_{k+2}]_0 = (k^2 - n^2)[y_k]_0.$$

$a^2 + b^2 = 1$ includes both cases $a = 1$, $b = 0$, and $a = 0$, $b = 1$, so y, which may be $\cos n\theta$ or $\sin n\theta$, can be expanded by Maclaurin's Theorem either in powers of $\cos\theta$ or in powers of $\sin\theta$. It is only necessary to find the first two terms, and then the relation between $[y_{k+2}]_0$ and $[y_k]_0$ determines the whole series.

For example, if $a = 0$, $b = 1$, $\therefore x = \sin\theta$; and when $\theta = 0$, $x = 0$, $y = A$, $\dfrac{dy}{dx} = nB$; thus

$$A\cos n\theta + B\sin n\theta$$

$$= A + nB\sin\theta + \sum_1^\infty (-1)^r\left\{A\frac{n^2(n^2-2^2)\ldots\{n^2-(2r-2)^2\}}{\lfloor 2r}\sin^{2r}\theta + B\frac{n^2(n^2-1^2)\ldots\{n^2-(2r-1)^2\}}{\lfloor 2r+1}\sin^{2r+1}\theta\right\}.$$

A less familiar result is as follows:

When n is half an odd integer, $A\cos n\theta + B\sin n\theta$ is

$$(-1)^{\frac{n^2}{2}-\frac{1}{8}}.2^{-\frac{1}{2}}\Big[\{A + (-1)^{n-\frac{1}{2}}B\} + \{(-1)^{n+\frac{1}{2}}A + B\}n\cos\theta + \sum_1^\infty (-1)^r\{A + (-1)^{n-\frac{1}{2}}B\}\frac{n^2(n^2-2^2)\ldots\{n^2-(2r-2)^2\}}{\lfloor 2r}\cos^{2r}\theta + \sum_1^\infty (-1)^r\{(-1)^{n+\frac{1}{2}}A + B\}\frac{n(n^2-1^2)\ldots\{n^2-(2r-1)^2\}}{\lfloor 2r+1}\cos^{2r+1}\theta\Big].$$

D. COXETER.

827. [X.2.] *Easy Calculation of Products.*

The following is a method (which I believe to be new) of calculating products exactly, with as much ease as the approximate results are arrived at by logs. and a trifle easier than those of the quarter square formula—which I am given to understand is considered the most efficient method at the moment.

Like both of these methods, my process requires a table of terms $\frac{1}{2}n(n+1)$, and for four-figured factors (*i.e.* eight-figure answers) just under 20,000 terms are required.

No.	Equivalent.	No.	Equivalent.
0	0	10	55
1	1	11	66
2	3	12	78
3	6	13	91
4	10	14	105
5	15	15	120
6	21	16	136
7	28	17	153
8	36	18	171
9	45	19	190

The rule is: Add the factors, and from the equivalent term of the total take the sum of the equivalent terms of the factors.

Decimal points are ignored until the answer is obtained, when the number of decimal places are calculated as in ordinary arithmetical operations.

			Equivalent terms.
Ex.	Reqd.	6×13	6.....21 +13.....+91 19.....190 − 112 = 78 *Ans.*
„	„	18×0	18.....171 +0.....+0 18.....171 − 171 = 0 *Ans.*
„	„	1×1	1.....1 +1.....+1 2.....3 − 2 = 1 *Ans.*
„	„	$\cdot04\times1\cdot3$	13.....91 +4.....∴...+10 17.....153 − 101 = 52 3 decimal places = ·052 *Ans.*

The system, as far as I know at present, *cannot* perform the other functions of logs.

I venture to suggest that it is an improvement upon the quarter-square formula (with which it is, however, related) for the two following reasons: (1) the fractional part of the tables is eliminated; (2) there is only one subtraction in my proposal as compared with two in the other. On the other hand, it is true that *three* (instead of two) equivalent terms must be abstracted from the tables—though, of course, the mental effort and time required for this extra term are very small. F. L. Hutchins.

828. [D. 2. a.] *To find upper and lower limits to the sums of the series formed by omitting from the series* $\Sigma 1/n$ *the fractions whose denominators contain* (*a*) *the digit* 0, (*b*) *the digit* 1. *v. Note* 662, May 1923, p. 311.

(*a*) The sum of the fractions whose denominators contain 2 digits is

$$\left(\frac{1}{11}+\ldots+\frac{1}{19}\right)+\left(\frac{1}{21}+\ldots+\frac{1}{29}\right)+\ldots+\left(\frac{1}{91}+\ldots+\frac{1}{99}\right),$$

and this is $<\frac{9}{10}+\frac{9}{20}+\ldots+\frac{9}{90}$ and $>\frac{9}{20}+\frac{9}{30}+\ldots+\frac{9}{100}$.

The fractions whose denominators contain 3 digits are

$$\left(\frac{1}{111}+\ldots+\frac{1}{119}\right)+\ldots+\left(\frac{1}{191}+\ldots+\frac{1}{199}\right)$$
$$+\left(\frac{1}{211}+\ldots+\frac{1}{219}\right)+\ldots+\left(\frac{1}{191}+\ldots+\frac{1}{199}\right)+\text{etc.};$$

and this is $<\frac{9^2}{100}+\frac{9^2}{200}+\ldots+\frac{9^2}{900}$ and $>\frac{9^2}{200}+\frac{9^2}{300}+\ldots+\frac{9^2}{1000}$;

similarly for the remaining terms.

Thus the sum of the series lies between

$$\left(1+\frac{1}{2}+\ldots+\frac{1}{9}\right)\left\{1+\frac{9}{10}+\left(\frac{9}{10}\right)^2+\ldots\right\}$$

and $$1+\frac{1}{2}+\ldots+\frac{1}{9}+\left(\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{10}\right)\left\{\frac{9}{10}+\left(\frac{9}{10}\right)^2+\ldots\right\},$$

i.e. between $10\left(1+\frac{1}{2}+\ldots+\frac{1}{9}\right)$ and $10\left(1+\frac{1}{2}+\ldots+\frac{1}{9}\right)-8{\cdot}1$,

i.e. between 28·29 and 20·18.

(*b*) The sum of the fractions whose denominators contain 2 digits is

$$\left(\frac{1}{20}+\frac{1}{22}+\ldots+\frac{1}{29}\right)+\left(\frac{1}{30}+\frac{1}{32}+\ldots+\frac{1}{39}\right)+\text{etc.},$$

which is $<\frac{9}{20}+\frac{9}{30}+\ldots+\frac{9}{90}$ and $>\frac{9}{30}+\frac{9}{40}+\ldots+\frac{9}{100}$, and so on.

The sum of the series therefore lies between

$$10\left(\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{9}\right) \text{ and } 10\left(\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{9}\right)-3{\cdot}6,$$

i.e. between 18·29 and 14·68.

In (*a*) the upper limit can be lowered to

$$1+\frac{1}{2}+\ldots+\frac{1}{9}+9\left(\frac{1}{1\cdot1}+\frac{1}{2\cdot1}+\ldots+\frac{1}{9\cdot1}\right),$$

which is less than 27.

For the lower limit, since

$$\frac{1}{a-b}+\frac{1}{a+b}>\frac{2}{a} \quad (a>b),$$

the 2-digit terms $$>\frac{9}{15}+\frac{9}{25}+\ldots+\frac{9}{95}.$$

Similarly the 3-digit terms

$$>\frac{9^2}{155}+\frac{9^2}{255}+\ldots+\frac{9^2}{955},$$

and the 4-digit terms $$>\frac{9^3}{1555}+\frac{9^3}{2555}+\ldots+\frac{9^3}{9555},$$

and so on.

Thus the whole series is greater than

$$1+\frac{1}{2}+\ldots+\frac{1}{9}+2\left[\frac{9}{10}\left(\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{19}\right)\right.$$
$$+\left(\frac{9}{10}\right)^2\left(\frac{1}{3\cdot1}+\frac{1}{5\cdot1}+\ldots+\frac{1}{19\cdot1}\right)$$
$$\left.+\left(\frac{9}{10}\right)^3\left(\frac{1}{3\cdot11}+\frac{1}{5\cdot11}+\ldots+\frac{1}{19\cdot11}\right)+\text{etc.}\right],$$

which is still greater than

$$1+\frac{1}{2}+\ldots+\frac{1}{9}+2\times9\left[\frac{1}{3\frac{1}{2}}+\frac{1}{5\frac{1}{2}}+\ldots+\frac{1}{19\frac{1}{2}}\right]$$

or $$2{\cdot}829+9^2\left(\frac{1}{14}+\frac{1}{23}+\frac{1}{32}+\ldots+\frac{1}{86}\right) \text{ or } 22{\cdot}8.$$

Thus series (*a*) lies between 22·8 and 27.

In (b) the lower limit can be raised in the same way

$$\left(\text{noting that } \frac{1}{10x}+\frac{1}{10x+9}>\frac{1}{10x+5}\right)$$

to $$\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{9}+18\left(\frac{1}{5\frac{1}{9}}+\frac{1}{7\frac{1}{9}}+\ldots+\frac{1}{19\frac{1}{9}}\right),$$

which is greater than 16.

Thus series (b) lies between 16 and 18·3. N. M. GIBBINS.

829. [D. 6. b. ; B. 1.] *The series for* $\log_e(1+ax+bx^2+cx^3+dx^4+\ldots)$.

If $1+p_1x+p_2x^2+\ldots=(1-a_1x)(1-a_2x)(1-a_3x)\ldots$, then, as given in Theory of Equations, we have:

$$\log(1+p_1x+p_2x^2+\ldots)=-s_1x-\tfrac{1}{2}s_2x^2-\tfrac{1}{3}s_3x^3-\ldots,$$

where $s_r=a_1{}^r+a_2{}^r+\ldots$.

The s's can be expressed as determinants in terms of the p's, and there is a quite elementary proof of this (see *e.g.* Smith's *Algebra*). It would seem worth while, then, for treatises on Algebra to give the interesting relation (in which for simplicity we write $p_1=a$, $p_2=b$, etc.),

$$\log(1+ax+bx^2+cx^3+dx^4+\ldots)$$

$$=ax-\tfrac{1}{2}\begin{vmatrix} a & 2b \\ 1 & a \end{vmatrix}x^2+\tfrac{1}{3}\begin{vmatrix} a & b & 3c \\ 1 & a & 2b \\ 0 & 1 & a \end{vmatrix}x^3-\tfrac{1}{4}\begin{vmatrix} a & b & c & 4d \\ 1 & a & b & 3c \\ 0 & 1 & a & 2b \\ 0 & 0 & 1 & a \end{vmatrix}x^4+\ldots,$$

which does not seem to be stated anywhere.

Along with this we have the corresponding relation

$$ax-\tfrac{1}{2}bx^2+\tfrac{1}{3}cx^3-\tfrac{1}{4}dx^4+\ldots$$

$$=\log\left\{1+ax+\frac{1}{2!}\begin{vmatrix} a & b \\ 1 & a \end{vmatrix}x^2+\frac{1}{3!}\begin{vmatrix} a & b & c \\ 2 & a & b \\ 0 & 1 & a \end{vmatrix}x^3+\frac{1}{4!}\begin{vmatrix} a & b & c & d \\ 3 & a & b & c \\ 0 & 2 & a & b \\ 0 & 0 & 1 & a \end{vmatrix}x^4+\ldots\right\}.$$

By working with known series numerous results in determinants may be obtained.

Application to the Multinomial Theorem.

The last equation can be applied to the multinomial theorem. It will be convenient to write the multinomial in the form $(x-\frac{1}{2}ax^2+\frac{1}{3}bx^3-\frac{1}{4}cx^4+\ldots)^n$.

The above expression is the coefficient of $\frac{y^n}{n!}$ in the expansion of

$$e^{y(x-\frac{1}{2}ax^2+\frac{1}{3}bx^3-\ldots)};$$

but the index here, viz. $yx-\frac{1}{2}ayx^2+\frac{1}{3}byx^3-\ldots$, is now seen to be the logarithm of

$$1+yx+\frac{1}{2!}\begin{vmatrix} y & ay \\ 1 & y \end{vmatrix}x^2+\frac{1}{3!}\begin{vmatrix} y & ay & by \\ 2 & y & ay \\ 0 & 1 & y \end{vmatrix}x^3+\frac{1}{4!}\begin{vmatrix} y & ay & by & cy \\ 3 & y & ay & by \\ 0 & 2 & y & ay \\ 0 & 0 & 1 & y \end{vmatrix}x^4+\ldots,$$

which is, therefore, the expansion of this power of e; and by picking out from the successive terms the coefficient of $\frac{y^n}{n!}$, we obtain the expansion of the multinomial.

While the multinomial theorem is inevitably very cumbersome, this process at any rate shows exactly what terms will occur in the expansion, and it probably supplies the easiest method for finding the coefficient of any given power of x. G. OSBORN.

The Leys, Cambridge.

REVIEWS.

(I.) An Introduction to Mathematical Probability. By J. L. COOLIDGE. Pp. xii + 216. 15s. 1925. (Oxford University Press.)

(II.) How to calculate Correlations. By G. H. THOMSON. Pp. 24. 2s. 1924. (Harrap.)

(III.) Vorlesungen über die Grundzüge der mathematischen Statistik. By C. V. L. CHARLIER. Pp. 125. n.p. 1920. (Verlag Scientia, Lund.)

(I.) A book on Probability by a mathematician of Professor Coolidge's eminence deserves careful study, even if the result is not entirely satisfying.

The preface makes brief mention of some of the difficulties of the subject, and states that: "Perhaps the most characteristic feature of the present work is that the statistical definition of probability is adhered to throughout." The basis of the work is set out in the first chapter, which will be considered presently. Chapters II. and III. are concerned with Elementary Principles and Bernoulli's Theorem, and Chapter V. with Geometrical Probability; these, with Chapter VI. on the contentious subject of Probability of Causes, are the portions in which we are in any way interested in probability as such.

In Chapters IV. and VII.-IX. we are more definitely concerned with statistical frequency. Those chapters deal with Mean Value and Dispersion, Errors of Observation, Errors in Many Variables, and Indirect Observations. The treatment of this portion is not as full as we could have wished. Correlation, for example, has only five pages; curve-fitting has seven, of which the last two are unnecessary. There are not many illustrative examples, but the book concludes with two welcome chapters on the Statistical Theory of Gases and the Principles of Life Insurance.

We must now return to Chapter I. The system there set out may be paraphrased as below. I replace "empirical assumption" by "axiom," which I think means exactly the same.

(1) *Axiom* 1. If in N happenings of an event, under the same essential conditions, it happens M times in way A and $N-M$ times not in way A, the ratio M/N, as N is indefinitely increased, will approach a definite limit.

(2) *Definition* of "probability." This limit is called the *probability* that the event will happen in way A, under those conditions.

(3) *Axiom* 2. If an event can happen in n ways, all of which are equally likely, and if we label m of these ways, the probability that the event will happen in one of these labelled ways is m/n.

(4) *Definition* of "equally likely." "If we can say that the total resultant of the essential conditions does not tend to favour one outcome rather than another, the two may be said to be *equally likely*."

(5) *Axiom* 3. If in (1) and (2) the way of happening of the event can be described by the values of certain variables, and if "way A" means that these values lie within the limits $X_1 \pm \frac{1}{2}dX_1$, $X_2 \pm \frac{1}{2}dX_2$, ... $X_k \pm \frac{1}{2}dX_k$, then the probability of way A is of the form $F(X_1X_2 \dots X_k)dX_1dX_2 \dots dX_k$, or differs from this by an infinitesimal of higher order.

(6) "Theorem of total probability, special case." (*Addition Theorem.*) "The probability that one of any number of mutually exclusive events should happen is the sum of the probabilities for the separate events."

(7) *Definition* of "independence." "Two events are said to be *mutually independent* when the probability for either is the same whether the other happen or not."

(8) "Theorem of compound probability." (*Multiplication Theorem.*) "If a compound event consists in the conjunction of any number of independent events, the probability of the compound event is the product of the probabilities for the individual events."

Professor Coolidge does not disguise the fact that this system is open to serious objection; indeed, he says plainly, in reference to (2), that "there are quite as many objections to this method of defining probability as to any other." But it is not clear that he has not increased the difficulties.

(i) Axiom 1 is at first sight harmless, even if we cannot accept the reasoning by which Professor Coolidge supports it. But, when we come to its applications, the serious question arises—what is meant by "approach"? By what route does the ratio approach the limit? Is the method of approach the same as that of a mathematical function? If so, does the author mean that, *assuming*—for we can never *know* what the probability of anything is—that the probability of drawing the ace of spades from a pack is 1/52, then, however small a fraction a may be, there is a number N_a such that we can be sure that in N_a or more trials the proportion of such drawings will always be within a of 1/52?

These questions are not hypercritical. The whole structure of the mathematical theory of probability is based on two propositions, usually known as the Addition Theorem and the Multiplication Theorem—(6) and (8) above. Whether they are properly to be regarded as theorems or as axioms is open to doubt; but they can be converted into theorems by a proper choice of axioms and definitions. In proof of the Addition Theorem, the author says that "one of the basic theorems of the infinitesimal calculus tells us that the limit of the sum of two variables dependent upon the same third variable is the sum of their limits." But this basic theorem is only established for a very precise meaning of the word "limit," which does not seem to hold here; the proof therefore seems invalid.

(ii) Apart from this mathematical difficulty, there is the psychological difficulty of basing a theory on a definition which involves an indefinitely great number; or, to put it differently, there is the objection to basing our present actions on what we assume will be the result of an eternity of experience.

This difficulty arises in looking at the proof of the Multiplication Theorem. The proof may be sound, or it may not; but the reader feels that there is a certain unreality in it.

(iii) and (iv). Axiom 2 seems to be quite unnecessary, as an axiom. The author says (p. 32) that "no mathematical formula can prove this assumption." Surely this is incorrect, with a proper interpretation of "equally likely." The natural interpretation is that two events are equally likely if their probabilities are equal. The "axiom" is then a very simple special case of the addition theorem; for, if the event can happen in n ways, all of which are equally likely, the probability of each way is $1/n$, and the probability of some one of m specified ways is m/n.

(v) The only point worth noting in reference to Axiom 3 is that the author integrates; thus, apparently quite unconsciously, using the addition theorem before he has proved it or even mentioned it.

As the author holds (p. vi) "that, in the last analysis, probability is a statistical, that is to say, an experimental science," it is rather a pity that he does not, especially in the later parts of the book, use the language of statistics, where it seems appropriate, rather than that of probability. As regards (7) and (8), for example, the statistical independence of two events is far easier to grasp than the statement that "Two events are said to be mutually independent when the probability for either is the same whether the other happen or not": and the multiplication theorem follows at once. (Indeed, the theorem is now very little more than a test of statistical independence.) And it is rather a shock to find (p. 60) that the definition of the mean value of a variable involves the "probabilities" of its different values. If the subject were treated statistically from the beginning, the whole of the mathematical theory of probability might be given in a single sentence: If the relative frequency of occurrence of an event E in circumstances C is p, then the probability of occurrence of E in circumstances C is p. But the time is hardly ripe for this; and, indeed, the word "probability" is often useful, even though the person who uses it is not thinking of probability at all.

Taken as a whole, the book appeals to the specialist who is interested in the theory of probability rather than to the beginner who wants to know something about statistical method. Unfortunately it has not been very carefully prepared for publication. It abounds in awkward phrasing, in omissions of important words, and in discrepancies of language; and some of

the algebra might quite well be simplified. There is no subdivision of chapters, except into large sections, each occupying about 6 to 8 pages; this is especially confusing, as the enunciation of a theorem sometimes precedes the proof, sometimes follows it. It may be added that the use of "precision" and of the form e^{-c^2} will be rather unwelcome to the English student, who is accustomed to work in terms of the dispersion (standard deviation).

(II.) Dr. Thomson has produced a very useful little guide to the calculation of coefficients of correlation. The coefficients mentioned are those of Spearman (rank), Pearson (product-moment and mean square contingency), and Yule (colligation). The formula for the probable error of the product-moment coefficient is also given.

(III.) This is a second edition of Dr. Charlier's work. It differs from the first edition mainly in the insertion of a supplementary chapter, dealing with (*a*) the calculation of the constants of a Gaussian distribution when the sorting is into three classes only; (*b*) Pearson's "tetrachoric" method of calculating a correlation-coefficient (very briefly treated); and (*c*) the method of arriving at what the author calls the "Bernoulli" correlation-coefficient, which is really Pearson's ϕ (square root of mean square contingency).

W. F. SHEPPARD.

Mécanique Statistique Classique. Tome II. Fascicule III. By ÉMILE BOREL. Pp. 146. Fr. 18. 1925. (Gauthier-Villars.)

This part of the *Calcul des Probabilités* is devoted to the development of the classical results of statistical mechanics. M. Borel has succeeded in showing how the ideas underlying the Theory of Probability naturally arise in mechanics, and how the fundamental laws of the Kinetic Theory are related to mechanical principles. A special feature of the book is the discussion of the establishment of a phase and a verification of the conservation of the volume of a phase in the case of a reflection in n dimensional space.

The law of the distribution of velocities in the steady state, and the proof of the decreasing nature of Boltzmann's H-Function and the related paradox of irreversibility, receive adequate and careful treatment. The author gives the generalisation of Maxwell's law for complex molecules, and obtains the Equipartition law; but he has evidently left the detailed discussion of the equipartition of energy to the section of the treatise which is to be devoted to the Quantum Theory.

The principles developed are applied to the determination of the pressure, the temperature, and the specific heat of a gas, and to the mean free path of a molecule. The relation between the entropy of a state and the probability of its occurrence is obtained in an interesting chapter on the Principle of Evolution.

The author does not give any discussion of the unsteady state, and is therefore unable to treat the phenomena of Viscosity and Thermal Conductivity. An interesting treatment of the Diffusion of Gases is based on the law of distribution of velocities in the steady state. This phenomenon, however, can only be discussed satisfactorily, if use is made of the results obtained for the distribution of velocities in the unsteady state. Although it is probably beyond the scope of the monograph to deal with the unsteady state, the addition of references to recent work on this part of the subject would greatly increase the value of the book.

Calcul des Probabilités. Tome I. Fascicule I. By ÉMILE BOREL. Pp. 155. Fr. 18. 1925. (Gauthier-Villars.)

This book forms Part I. of the first of four volumes of a *Traité du Calcul des Probabilités et de ses applications*, which the eminent mathematician, M. Borel, has undertaken to edit. The book is written with the lucidity and accuracy which characterises the author's mathematical writing, and should prove a notable contribution to the literature on the Theory of Probability. It will prove of service to those who are interested in the theoretical part of the subject as well as to those primarily concerned with the applications of the theory to physical science.

One hundred pages of the book are devoted to the development of the elementary results, and M. Borel illustrates his theorems by considering the problems which arise in the throwing of dice, the tossing of coins, and the drawing of coloured balls from urns. A special feature of the book is the skilful geometrical representations which are given, and which result in a considerable simplification of the analysis required to deal with the probability that the distance between a pair of points on a line, in a plane, or in space, should be less than an assigned value.

In the development of Gauss' Law of Error and Stirling's formula for the asymptotic value of $n!$, the author's elegant treatment of analytical problems is much in evidence. The principles underlying the theory of correlation are clearly set forth, but it is to be hoped that a more detailed account of this part of the subject is to be given in another part of the treatise.

In the last fifty pages of the book, M. Borel discusses Statistics. Here his main objective is to examine the fundamental problem which confronts the pure mathematician, viz. to deduce the law of error in a phenomenon for which the errors, and the first $2n$ moments of the errors, are given. M. Borel does not give Tchébychef's solution, but simplifies the problem and gives a masterly exposition of the work of Stieltjes. The theorem which M. Borel proves may be stated as follows: Given an arbitrary function $f(x)$, which is increasing in a finite positive interval and which has the same first $2n$ moments as a discontinuous function represented by n steps of a staircase, then the discontinuous function is a first approximation to $f(x)$ and becomes $f(x)$ when n is increased indefinitely.

M. René Lagrange contributes a resumé of the work of Tchébychef on mean values and the problem of moments, together with a note on the polynomials of Hermite-Tchébychef. A bibliography of recent work on the problem of moments and mean values enhances the value of the monograph.

The Principles of Thermodynamics. By GEORGE BIRTWISTLE. Pp. 163. 7s. 6d. 1925. (Cambridge University Press.)

This book contains "the substance of lectures given in the University of Cambridge to men whose future interest may have been any of mathematics, physics, chemistry, astronomy, or mechanical science." The author's purpose is to set out with care the fundamental principles of the subject, and to illustrate them by applications to various branches of science. This purpose has been achieved. The book is a most valuable elementary treatise on Thermodynamics. The author has avoided overloading the book with formulae and has emphasised the thermodynamical principles involved. The value of the work is enhanced by historical notes and by references to original papers. A special feature is the discussion of parts of the subject which are receiving much attention at the present time. A discussion of Planck's radiation formula, of Debye's theory of specific heat at low temperatures and of modern electron theory, is particularly welcome in an elementary work.

The principles are applied to the discussion of Rankine's cycle, the phase rule, osmotic pressure, thermo-electric phenomena, radiation, and stellar spectra.

The author has not referred to the relation of statistical mechanics to thermodynamics, on the ground that a special knowledge of dynamical theory is required for this purpose. This is probably true, but the admirable skill shown by the author in presenting thermodynamical principles leads one to hope that this part of the subject is being deferred for special treatment.

The book will prove of great service to those who desire an exact and interesting introduction to the study of thermodynamics. J. MARSHALL.

Exercises in Geometry. Part I. By V. LE NEVE FOSTER. Pp. viii + 69. 2s. 1925. (Bell & Sons.)

Mr. Foster has already brought out a treatise on Plane and Solid Geometry, in three vols., all very favourably reviewed in No. 163 of the *Gazette*. This book of exercises has the same judicious use of practical work as an introduction, and a constant accompaniment, to theory, and will be welcome to other teachers besides those who use his text-book. As he says, and as

teachers of experience will agree, "a collection of riders can hardly be too big." The exercises are in four sets, A, B, C, D; A and B, practical and theoretical respectively, corresponding to Vol. I. of the text-book; C and D in the same way to Vol. II. They are of a greatly varied character, and may be taken in any order convenient to the teacher who uses it. They are illustrated by a set of excellent diagrams, many of them of an arresting character, due to the use of continuously blacked spaces. Some of these remind us of Moorish geometrical patterns, which might indeed often be profitably used by teachers for stimulating the interest of their pupils in accurate draughtsmanship. Pages 1 and 2 contain a set of aphorisms, giving sound advice to pupils who want to get full credit for good work, but are apt to lose it through bad arrangement or careless omissions; *e.g.* "It is not his (the Examiner's) business to guess, or to spend time in trying to find out, what you may have meant, or to decipher difficult handwriting." "It is helpful, when possible, to use the senses of 'touch' and 'sight' to assist the brain. [In Solid Geometry makeshift models and a little imagination are very useful.]"

This mention of Solid at the beginning of the book caused a careful but unsuccessful search through the exercises to see if among them we could find a fair number of a much wanted kind. There are, it is true, examples on the shadows of sticks, on angles subtended at the eye by diameters of distant discs, on the area of the surface of water in a partially filled hemispherical bowl, but they are all such as can be solved with the necessary lines actually in the plane of the paper. We hold very strongly the view of the Committee, given on p. 22 of the *Report on the Teaching of Geometry in Schools*, "that there ought to be no formal separation between plane and solid geometry," and that the author's *pari passu* should have applied to this companionship, as well as to that of practice and theory. We urge the addition of a considerable number of very elementary exercises on Solid, *e.g.* such as easy ones on finding heights and distances, when a new edition is wanted, which shall lead on to others, increasing in number and difficulty, in the similar collections for later stages of work which the author seems to have in readiness for publication. We hope that the reception of Part I. may be such as to ensure the early appearance of its successors.

Geometry for Beginners. By J. G. Bradshaw. Pp. ii + 99. 2s. 6d. net. 1925. (Longmans.)

This takes its readers as far as the Theorem of Pythagoras. Section I. (1-57) is intended as a basis for oral teaching, and supplies a useful series of graduated exercises, through which pupils may show, and teachers may estimate, to what extent this has been effective. Section II. (58-97) is devoted to formal theorems and constructions. To about half of these proofs are supplied to serve as models of style and arrangement, the others having little beyond enunciation and figure with a hint.

It shows signs of experience in teaching, and seems well adapted for its purpose. Type, figures and arrangement are good.

Edward M. Langley.

Principles of Geometry. By H. F. Baker. Vol. IV. Higher Geometry: being illustrations of the utility of the consideration of higher space, especially of four or five dimensions. Pp. 250. 15s. net. 1925. (Cambridge University Press.)

The main theme of this volume is that figures in space of any number of dimensions can be deduced, by projection or some such process, from much simpler figures in space of higher dimensions, of which four are useful as coming next after three, and five as affording the simplest representation of the coordinates of a line.

After a survey of the field, there follows a treatment of plane theorems, on systems of circles, and the plane quartic of genus 1, obtained by considerations in ordinary space. Line geometry is derived from five dimensions, where Veronese's surface arises in connection with the chords of a cubic. In four dimensions, Segre's figure of fifteen lines and points is studied; and the

cyclides are deduced from a quartic surface analogous to the quartic curve of intersection of two ordinary quartics, but simpler in that it is rational. Professor Baker holds this to be the simplest approach to the theory of the cubic surface. The last chapter gives a similar treatment of Kummer's surface.

Whether an approach from higher space can be the simpler for any particular student, depends on his previous training: most of us will never be as much at home in five-fold space as the distinguished author. But even to one hindered by a strong habit of visualisation and reliance on ordinary intuitions, his point of view is fascinating and his illustrations convincing.

H. P. H.

Linear Integral Equations. By W. V. Lovitt, Ph.D. Pp. xiii + 253. 15s. net. 1924. (M'Graw-Hill Book Company.)

This book deals nearly exclusively with the functional equation of the Fredholm type,

$$u(x) = f(x) + \lambda \int_a^b k(x, t) u(t)\, dt,$$

where $f(x)$ and $k(x, t)$ are known functions, a and b are constants and λ is a constant parameter, and the problem is to find a function $u(x)$ which satisfies the equation. The author confines himself to the original, now classical, methods. He devotes Chapter II to Volterra's equation, where b, the upper limit of integration, is replaced by x; this is the particular case of Fredholm's equation, for which $k(x, t) = 0$ when $t > x$, and it admits of special treatment. There follow two chapters in which are given Fredholm's theory *in extenso*, and applications of it to some Dirichlet and Neumann boundary problems. This accounts for about one-half of the book. The remainder is devoted to the Hilbert-Schmidt solution of the equation for the symmetrical case, that is the case where $k(x, t) \equiv k(t, x)$, and some of Hilbert's applications.

The equation of Fredholm summarises in a most remarkable way a great part of the boundary problems of mathematical physics, and twenty years ago, when a complete formal solution was discovered by Fredholm and a different solution for the symmetric case by Hilbert, there were high hopes that the new theory would lead to important new developments in this science. These hopes were deceived, partly no doubt because the boundary problems had already been so industriously explored by means of differential equations. Further disappointment was felt when no practicable means was devised of producing numerical results from the formal solutions. But the main reason for the comparative failure of the theory in this direction was the rise of atomic physics on the one hand and of relativity and the tensor calculus on the other. The claim of Fredholm's equation upon the physicist remains a strong one, however, for by virtue of its simplicity, elegance and great generality it has displaced differential equations as the standard method of the old mathematical physics.

Its claim upon the mathematician is greater, for it is closely bound up with orthogonal and bi-orthogonal functions, a subject which has been considerably developed, and further with some very interesting classes of transformations.

Dr. Lovitt adheres closely to the classical memoirs, and reproduces the theory of the subject as it stood about the year 1908. There is thus nothing novel either in the substance or the presentation, and the book might equally well have been produced at or soon after this date. Accepting these limitations, we will hasten to say that the author has done his work well. He has known how to select the essentials from the classical theories, and to reproduce them simply and clearly in a uniform notation without ever departing far from the original. The book may be recommended to students who are starting the subject, and who will be saved thereby some laborious reading.

It is usual in English college classes, and to a less degree in American to refrain from teaching anything that has been discovered within the last twenty years. From this point of view Mr. Lovitt's book is timely and appropriate, for apart from tracts like that of Professor Bôcher, I believe this is the first English text-book on the subject.

At the same time I may perhaps point out that there would have been distinct advantages in utilising more modern ideas. In the matter of integration, for instance, although Mr. Lovitt does not say what he means by an integral, we may assume that he intends the Riemann or the Cauchy definition. This leads him, in discussing the scope and validity of his results, to postulate continuity in one form or another. Now continuity is not a natural limitation to integration, and its application is often both awkward and tiresome. Had a more modern definition of the integral been adopted, these questions of scope and validity would have been very greatly simplified without in any way affecting the applications. This might, however, have alarmed readers, and possibly have discouraged lecturers from basing courses upon the book.

One direction in which the book might be improved is in the welding together of the Fredholm and Hilbert theories. These are treated almost independently. But there would be a stronger sense of unity as well as some saving of effort if more use were made in the later chapters of the results in the earlier ones. For instance, Theorem II on page 120 follows from Theorem III on page 19, and equation (18) on page 34. Kneser's proof, which is reproduced on pp. 118-120, was published at a time when the Fredholm results were not generally familiar.

The cross-word definition of an optimist is a man who writes in his clues in ink. Dr. Lovitt, I think, is one of them, for he prints "First Edition" on the title-page. I hope his optimism may be justified. H. B. HEYWOOD.

The Elements of Mechanics. By F. S. CAREY and I. PROUDMAN. With Diagrams. Pp. i-ix, Text 1-188, Examples 191-304, Answers and Index 305-314. 1925. (Longmans, Green & Co.)

The book consists of a summary of elementary mechanical principles up to and including the dynamics of a rigid body, the Stroud system being adopted, in which symbols denote quantities and not mere numbers. It is not in a form which would be assimilated by a beginner, but could be read by way of revision and coordination of ideas.* There are two short chapters on elementary Hydrostatics, and others on the history and principles of Mechanics, but too brief to be of great interest. The book concludes with an excellent collection of examples arranged in accordance with the successive chapters of the summary, typical examples being worked out in connection with each chapter. Answers to the examples are given at the end. A. LODGE.

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* The ideas and methods of vector addition are explained and used, and the ideas of the vector and scalar products of two vectors are employed in connection with *moments* and *work* respectively. A knowledge of the Infinitesimal Calculus is not assumed.

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